AN APPROXIMATE APPROACH TO FRACTIONAL STOCHASTIC INTEGRATION AND ITS APPLICATIONS

NGUYEN TIEN DUNG AND TRAN HUNG THAO

Abstract. The aim of this paper is to introduce an approximation approach to fractional stochastic integration. Based on obtained result we find explicit solution of some fractional stochastic differential equations and study the ruin probability in the ALM model.
2000 AMS Classification: 26A33, 91B30.
Key words: fractional Brownian motion, ruin probability, asset liability management.

1. Introduction

The fractional Brownian motion (fBm) and the related problems have been investigated by several authors from different approaches [1, 3, 4, 7, 8, 9]. A fBm with Hurst parameter $H \in (0,1)$ is a Gaussian process $W^H = \{W_t^H, 0 \le t \le T\}$ with $E[W_t^H] = 0$ and the covariance function $R_H(t,s) = E[W_t^H W_s^H]$ defined as

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\,. \tag{1.1}$$

for all $t, s \in [0,T]$.
In [9] Mandelbrot has given a representation of W^H of the form:

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\big[U_t + \int_0^t (t-s)^\alpha dW_s\big]\,, \tag{1.2}$$

where W is a standard Brownian motion, $\alpha = H - \frac{1}{2}$ and $U_t = \int_{-\infty}^0 \big((t-s)^\alpha - (-s)^\alpha\big) dW_s$. The process $\{U_t, 0 \le t \le T\}$ is of absolutely continuous trajectories. It is know that the second term of (1.2) is the main part expressing the long memory of W_t^H and is called a fractional Brownian motion of Liouville form [7, 11, 3].

In this paper, we consider the fBm of Liouville form with parameter $H \in (\frac{1}{2}, 1)$

$$B_t = \int_0^t (t-s)^{\alpha} dW_s \,. \tag{1.3}$$

In [1] E.Alòs, O.Mazet and D.Nualart used Mallivin Calculus method to approximate B_t by semimartingales B_t^{ε} defined as follows:

$$B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha} dW_s \ , \ \varepsilon > 0 \,. \tag{1.4}$$

T.H.Thao [11] has proved that $B_t^{\varepsilon} \xrightarrow{L^2(\Omega)} B_t$ when $\varepsilon \to 0^+$, the convergence is uniform with respect $t \in [0, T]$. Using Thao's results, we prove that if the stochastic process $f \in \bigcup\limits_{\mu > \frac{3}{4}} C^{\mu}[0, T]$ then

$$\int_0^t f(s) dB_s^{\varepsilon} \xrightarrow{L^1(\Omega)} \int_0^t f(s) d\mathrm{B}_s \ \ \forall \ t \in [0, T] \,.$$

where the left hand side is integral driven by a semimartingale, the right hand side is pathwise integral, which is constructed by Zähle [14, 15]. We use the obtained results to study some fractional stochastic differential equation and ruin probability in the ALM model.

2. Preliminaries

For the sake of convenience, we recall an important result, which will be the basis of this paper. For every $\varepsilon > 0$ we define

$$B_t^{\varepsilon} = \int_0^t (t-s+\varepsilon)^{\alpha} dW_s \quad , \ \ \alpha = H - \frac{1}{2} \,. \tag{2.1}$$

We have the following theorem:

Theorem 2.1. *Let $H \in (0, 1)$. Then*
I. *The process $\{B_t^{\varepsilon}, 0 \leq t \leq T\}$ is a semimartingale. Moreover*

$$B_t^{\varepsilon} = \int_0^t \varphi_{\varepsilon}(s) ds + \varepsilon^{\alpha} W_t \,, \tag{2.2}$$

where $\varphi_\varepsilon(t) = \alpha \int\limits_0^t (t-u+\varepsilon)^{\alpha-1} dW_u$.

II. *The process* B_t^ε *converges to* B_t *in* $L^2(\Omega)$ *when* ε *tend 0. This convergence is uniform with respect to* $t \in [0,T]$.

Proof. The detailed proof of this theorem can be found in [11].case $H > \frac{1}{2}$, we suggest a different way as follows:

For all $a, b > 0$ we have following inequality

$$(a+b)^\alpha \leq a^\alpha + b^\alpha \ \ \forall \ \ \alpha \in [0,1] . \tag{2.3}$$

Applying this inequality to $a = t-u, b = \varepsilon$ we obtain

$$E|B_t^\varepsilon - B_t|^2 = E\big(\int\limits_0^t [(t-u+\varepsilon)^\alpha - (t-u)^\alpha] dW_u \big)^2$$

$$= \int\limits_0^t [(t-u+\varepsilon)^\alpha - (t-u)^\alpha]^2 du \leq \int\limits_0^t \varepsilon^{2\alpha} du = \varepsilon^{2\alpha} t \ \forall \ t \in [0,T] .$$

So

$$E|B_t^\varepsilon - B_t|^2 \leq T\varepsilon^{2\alpha} \ \ \forall \ \ t \in [0,T] . \tag{2.4}$$

□

Remark 2.1. We recall some results on a generalization of the Stieltjes integral introduced by Zähle [14].
(i) We have the following estimate for all $t \in [0,T]$

$$| \int\limits_0^t f dg | \leq C(\lambda) \, \|f\|_{\lambda,1} \, \|g\|_{1-\lambda,\infty} . \tag{2.5}$$

(ii) If $f \in C^\lambda[0,T]$ and $g \in C^\mu[0,T]$ with $\lambda + \mu > 1$, it is proved that the integral $\int\limits_0^t f dg$ coincides with the Riemann-Stieltjes integral.

3. Main results

Proposition 3.1. *Suppose that* $H \in (\frac{1}{2}, 1)$ *and* $\varepsilon \in (0,1)$.

(a) *The following estimates hold for all* $t, s \in [0, T]$

$$E|B_t - B_s|^2 \leq c_1|t-s|^{2H} \tag{3.1}$$

$$E|B^{\varepsilon}_t - B^{\varepsilon}_s|^2 \leq c_1|t-s| \tag{3.2}$$

where c_1 *is a positive constant, depends only on* H *and* T.

(b) *Put* $D^{\varepsilon}_t = B^{\varepsilon}_t - B_t$ *then*

$$E|D^{\varepsilon}_t - D^{\varepsilon}_s|^2 \leq c_2\, \varepsilon^{\alpha}|t-s|^{1/2} \quad \forall \ t, s \in [0, T]\,. \tag{3.3}$$

where c_2 *is some constant depending only on* H *and* T.

(c) *For all* $0 < \lambda < \frac{1}{4}$ *we have following estimate*

$$E\|B^{\varepsilon} - B\|_{\lambda,1} \leq c_3 \varepsilon^{\frac{\alpha}{2}}\,.$$

where c_3 *depends only on* H, T *and* λ.

Proof. (a) The inequality (3.1) is elementary property of fBm and its proof can be found in [7]. The inequality (3.2) can be proved as follows:

Without loss of generality we may assume that $s \leq t$. By virtue of Itô integration we see that

$$E|B^{\varepsilon}_t - B^{\varepsilon}_s|^2 = \int_0^t (t-u+\varepsilon)^{2\alpha}du + \int_0^s (s-u+\varepsilon)^{2\alpha}du \\ - 2\int_0^s (t-u+\varepsilon)^{\alpha}(s-u+\varepsilon)^{\alpha}du.$$

We consider the right hand side of latest equality as a function in $t \in [s, T]$, denote by $f(t)$. It is clear that $f \in C^{\infty}[s, T]$ and we have $f(s) = 0, f'(s) = \varepsilon^{2\alpha}$, $f'(t) = (t+\varepsilon)^{2\alpha} - 2\alpha \int_0^s (t-u+\varepsilon)^{\alpha-1}(s-u+\varepsilon)^{\alpha}du$. Therefore

$$|f'(t)| \leq (t+\varepsilon)^{2\alpha} + 2\alpha\int_0^s (s-u+\varepsilon)^{2\alpha-1}du \\ = (t+\varepsilon)^{2\alpha} + (s+\varepsilon)^{2\alpha} - \varepsilon^{2\alpha} \leq 2(T+1)^{2\alpha} \quad \forall \ s \leq t \leq T\,.$$

The theorem of finite increment applied to the function $f(t)$ yields

$$f(t) = |f(t) - f(s)| \leq 2(T+1)^{2\alpha}\,|t-s|\,.$$

(b) Using (2.4) we obtain

$$E|D^{\varepsilon}_t - D^{\varepsilon}_s|^2 \leq 2\big(E|D^{\varepsilon}_t| + E|D^{\varepsilon}_s|\big) \leq 4T\,\varepsilon^{2\alpha}\,. \tag{3.4}$$

On the other hand,

$$
\begin{aligned}
E|D_t^\varepsilon - D_s^\varepsilon|^2 &\le 2\big(E|B_t^\varepsilon - B_s^\varepsilon|^2 + E|B_t - B_s|^2\big) \\
&\le 2c_1(1+(2T)^{2H-1})|t-s|\,.
\end{aligned}
\tag{3.5}
$$

Combining (3.4) and (3.5) we obtain

$$
E|D_t^\varepsilon - D_s^\varepsilon|^2 \le c_2\,\varepsilon^\alpha |t-s|^{1/2}\,.
$$

with $c_2 = \sqrt{8Tc_1((2T)^{2H-1}+1)}$.
(c) We have

$$
E\|B^\varepsilon - B\|_{\lambda,1} = \int_0^T \frac{E|D_s^\varepsilon|}{s^\lambda} ds + \int_0^T \int_0^s \frac{E|D_s^\varepsilon - D_y^\varepsilon|}{(s-y)^{\lambda+1}} dyds.
$$

Using the estimates (2.4) and (3.3) we obtain

$$
\begin{aligned}
E\|B^\varepsilon - B\|_{\lambda,1} &\le \int_0^T \tfrac{\varepsilon^\alpha\sqrt{T}}{s^\lambda} ds + \int_0^T\int_0^s \tfrac{\sqrt{c_2\,\varepsilon^\alpha\sqrt{(s-y)}}}{(s-y)^{\lambda+1}} dyds \\
&\le \varepsilon^{\alpha/2}\Big(\int_0^T \tfrac{\sqrt{T}}{s^\lambda} ds + \int_0^T\int_0^s \tfrac{\sqrt{c_2}}{(s-y)^{\lambda+3/4}} dyds\Big)\,.
\end{aligned}
$$

The integrals in the right hand side is finite because of $0<\lambda<1/4$.

Thus, there exists a finite constant c_3 such that

$$
E\|B^\varepsilon - B\|_{\lambda,1} \le c_3 \varepsilon^{\frac{\alpha}{2}}\,.
$$

The proof is complete.

□

The next theorem is a basic result of this paper.

Theorem 3.1. *Suppose that stochastic process $f : [0,T]\times\Omega \to \mathbb{R}$, satisfies the following condition: For some $\delta>0$ there exists a finite constant $M>0$ such that*

$$
\|f\|_{C^{\frac{3}{4}+\delta}[0,T]} < M \quad a.s.
$$

Then for all $t\in[0,T]$

$$
\int_0^t f(s)dB_s^\varepsilon \xrightarrow{L^1(\Omega)} \int_0^t f(s)dB_s \quad as\ \varepsilon \to 0\,.
$$

Proof. We first recall that

$$E|B_t - B_s|^2 \leq c_1|t-s|^{2H}$$

for all $t, s \in [0,T]$. As a consequence, the process B has β-Hölder continuous path for all $0 < \beta < H$, i.e: $B \in \bigcap\limits_{0<\beta<H} C^{\beta}[0,T]$ with probability one. Moreover, the process $f \in C^{\frac{3}{4}+\delta}[0,T]$. It follows from Remark 2.1 that integral $\int\limits_0^t f(s)d\mathrm{B}_s$ can be understood as a Riemann-Stieltjes integral. An application of the integration by parts formula to both integrals $\int\limits_0^t f(s)d\mathrm{B}_s$ and $\int\limits_0^t f(s)dB^{\varepsilon}_s$ we obtain

$$\int\limits_0^t f(s)dB^{\varepsilon}_s - \int\limits_0^t f(s)d\mathrm{B}_s = \int\limits_0^t f(s)d(B^{\varepsilon}_s - B_s)$$
$$= f(t)(B^{\varepsilon}_t - B_t) - \int\limits_0^t (B^{\varepsilon}_s - B_s)df(s).$$

Hence

$$\Big|\int\limits_0^t f(s)dB^{\varepsilon}_s - \int\limits_0^t f(s)d\mathrm{B}_s\Big| \leq |f(t)(B^{\varepsilon}_t - B_t)| + \Big|\int\limits_0^t (B^{\varepsilon}_s - B_s)df(s)\Big|.$$

Applying the Hölder inequality and the relation (2.4) yields

$$E|f(t)(B^{\varepsilon}_t - B_t)| \leq \|f(t)\|\,\|B^{\varepsilon}_t - B_t\| \leq M\sqrt{T}\varepsilon^{\alpha} \tag{3.6}$$

where $\|.\|$ stands for the $L^2(\Omega)$-norm.

Moreover, since $f \in C^{\frac{3}{4}+\delta}[0,T] \subset W^{\frac{3+\delta}{4},\infty}[0,T]$ a.s, we can apply the inequality (2.5) to $\lambda = \frac{1-\delta}{4}$ and obtain

$$\Big|\int\limits_0^t (B^{\varepsilon}_s - B_s)df(s)\Big| \leq C(\delta)\|B^{\varepsilon} - B\|_{\frac{1-\delta}{4},1}\|f\|_{\frac{3+\delta}{4},\infty}.$$

An easy computation leads us the inequality

$$\|f\|_{\frac{3+\delta}{4},\infty} \leq M(1 + \frac{4}{3\delta})\,T^{\frac{3\delta}{4}} := M_1 \text{ a.s.}$$

Hence from Proposition 3.1 we see that

$$E\Big|\int_0^t (B_s^\varepsilon - B_s)df(s)\Big| \le M_1 C(\delta)\, c_3\, \varepsilon^{\frac{\alpha}{2}}.$$

Combining the this inequality and (3.6) we have

$$E\Big|\int_0^t f(s)dB_s^\varepsilon - \int_0^t f(s)d\mathrm{B}_s\Big| < c_4 \varepsilon^{\frac{\alpha}{2}}\,.$$

with $c_4 = M\sqrt{T} + M_1 C(\delta)\, c_3$.
The theorem is thus proved. □

4. An application to fractional stochastic differential equation

In this section, we will give the explicit solution for an important class of fractional stochastic differential equations of the form

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma\, dB_t \\ \\ X_t|_{t=0} = X_0 \end{cases} \tag{4.1}$$

where $b(t,x) = \alpha(t)x + \beta(t)$ and σ is the known constant. We assume that $\alpha(t), \beta(t)$ are deterministic functions on $[0,T]$ and $\alpha(t)$ is bounded function by some constant $K > 0$.

A classical example of equations (4.1) is the Langevin equation

$$dX_t = (\alpha - bX_t)\, dt + \sigma\, dB_t.$$

We consider the corresponding approximation equation

$$\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon)\, dt + \sigma\, dB_t^\varepsilon \\ \\ X_t^\varepsilon|_{t=0} = X_0 \end{cases} \tag{4.2}$$

By extending a result by T.H.Thao and T.T.Nguyen [13] we will prove that the solution of the equation (4.1) is the limit in $L^1(\Omega)$ of the solution of (4.2) as ε tends 0.

Proposition 4.1. *Suppose that $H \in (\frac{1}{2}, 1)$. Then the solution X_t^ε of the equation (4.2) converges to the solution X_t of (4.1) in $L^1(\Omega)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$.*

Proof. We have

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b(s, X_s^\varepsilon)\, ds + \sigma \int_0^t dB_s^\varepsilon\,,$$

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \sigma \int_0^t dB_s$$

then

$$|X_t^\varepsilon - X_t| \leq \int_0^t |b(s, X_s^\varepsilon) - b(s, X_s)|\, ds + \sigma |B_t^\varepsilon - B_t|$$

$$\leq \int_0^t |\alpha(s)(X_s^\varepsilon - X_s)|\, ds + \sigma |B_t^\varepsilon - B_t|\,.$$

Using the estimate (2.4) and since $|\alpha(s)| \leq K\ \forall\ s \in [0, T]$ we obtain

$$E|X_t^\varepsilon - X_t| \leq K \int_0^t E|X_s^\varepsilon - X_s|\, ds + \sigma\sqrt{T}\varepsilon^\alpha \tag{4.3}$$

for all $0 \leq t \leq T$. A standard application of Gronwall's lemma starting from (4.3) will give us:

$$E|X_t^\varepsilon - X_t| \leq \sigma\sqrt{T}\varepsilon^\alpha e^{Kt}.$$

It follows that

$$\sup_{0\leq t\leq T} E|X_t^\varepsilon - X_t| \leq \sigma\sqrt{T}\varepsilon^\alpha e^{KT}.$$

So $X_t^\varepsilon \xrightarrow{L^1(\Omega)} X_t$ when $\varepsilon \to 0^+$, the convergence is uniform with respect $t \in [0, T]\,.$ ☐

Next, we will find the explicit solution of the equation (4.2)

Proposition 4.2. *Suppose that $H \in (\frac{1}{2}, 1)$ and X_0 is measurable random variable. Then the solution of (4.2) is given by*

$$X_t^\varepsilon = e^{\int_0^t \alpha(u)du}\Big(X_0 + \int_0^t \beta(s) e^{-\int_0^s \alpha(u)du} ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)du} dB_s^\varepsilon\Big).$$

Proof. By (2.2) we can rewrite the equation (4.2) in the following form

$$dX_t^\varepsilon = (\alpha(t)X_t^\varepsilon + \beta(t) + \sigma\varphi_\varepsilon(t))\,dt + \sigma\varepsilon^\alpha\,dW_t. \tag{4.4}$$

We split (4.4) into two equations:

$$dX_1(t) = (\alpha(t)X_1(t) + \beta(t))\,dt + \sigma\varepsilon^\alpha\,dW_t\,, \tag{4.5}$$

$$dX_2(t) = (\alpha(t)X_2(t) + \sigma\varphi_\varepsilon(t))\,dt. \tag{4.6}$$

The solution of (4.2) will be $X_t^\varepsilon = X_1(t) + X_2(t)$. We see that (4.5) is an Itô stochastic differential equation and its solution is given by:

$$X_1(t) = e^{\int_0^t \alpha(u)du}\big(X_1(0) + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)du}ds + \sigma\varepsilon^\alpha \int_0^t e^{-\int_0^s \alpha(u)du}dW_s\big).$$

The equation (4.6) is an ordinary differential equation for every fixed ω and its solution is:

$$X_2(t) = e^{\int_0^t \alpha(u)du}\big(X_2(0) + \sigma\int_0^t \varphi_\varepsilon(s)e^{-\int_0^s \alpha(u)du}ds\big).$$

Thus, noting that $dB_s^\varepsilon = \varphi_\varepsilon(s)ds + \varepsilon^\alpha dW_s$, the solution of (4.2) is

$$\begin{aligned} X_t^\varepsilon &= X_1(t) + X_2(t) \\ &= e^{\int_0^t \alpha(u)du}\big(X_0 + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)du}ds + \sigma\int_0^t e^{-\int_0^s \alpha(u)du}dB_s^\varepsilon\big). \end{aligned}$$

The proposition is proved. □

The next theorem is the most important result of this section.

Theorem 4.1. *Suppose that $H \in (\frac{1}{2}, 1)$ and X_0 is a random variable such that $E|X_0| < \infty$. Then the solution of (4.1) is the unique and given by*

$$X_t = e^{\int_0^t \alpha(u)du}\big(X_0 + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)du}ds + \sigma\int_0^t e^{-\int_0^s \alpha(u)du}dB_s\big).$$

Proof. First of all, by Proposition 4.1 and Proposition 4.2 we have only to prove that

$$\int_0^t f(s)dB_s^\varepsilon \xrightarrow{L^1(\Omega)} \int_0^t f(s)d\mathrm{B}_s \;\; \forall\, t \in [0, T]\,. \tag{4.7}$$

where $f(s) = \exp(-\int_0^s \alpha(u)du)$. This is obvious since $f \in C^1[0,T]$.

The uniqueness of the solution of (4.1) follows from that of L^1-limit: If $X_t^{(1)}$ and $X_t^{(2)}$ are limits of X_t^ε in $L^1(\Omega)$, then

$$E|X_t^{(1)} - X_t^{(2)}| \leq E|X_t^{(1)} - X_t^\varepsilon| + E|X_t^{(2)} - X_t^\varepsilon| \to 0 \text{ as } \varepsilon \to 0.$$

This complete the proof. □

5. The ruin probability in the ALM model

In finance and economics, it is usual to model the evaluation of the assets and the liabilities of a bank or of an insurance company with the use of stochastic processes for both parts of the balance sheet. This leads to useful models used in theory and practice of asset liability management (ALM).

In this section, we consider ALM model that the stochastic of the asset X_t and the liability Y_t satisfy the following stochastic differential equations

$$\begin{cases} dX_t = \mu_1 X_t dt + \sigma_1 X_t dB_t^{(1)} \\ dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dB_t^{(2)} \\ X|_{t=0} = X_0 \ , \ Y|_{t=0} = Y_0 < X_0 \end{cases} \tag{5.1}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are non-negative parameters.
$B_t^{(1)} = \int_0^t (t-s)^\alpha dW_s^{(1)}$, $B_t^{(2)} = \int_0^t (t-s)^\alpha dW_s^{(2)}$ being two fractional Brownian motions with correlation coefficient $|\rho| \leq 1$.

It is known from [12, 11] that the solution of the fractional stochastic differential equation in the fractional Black-Scholes model

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

is $S_t = S_0 e^{\mu t + \sigma B_t}$. It follows from this fact that

$$X_t = X_0 e^{\mu_1 t + \sigma_1 B_t^{(1)}} \ , \ Y_t = Y_0 e^{\mu_2 t + \sigma_2 B_t^{(2)}}$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp\left((\mu_1 - \mu_2)t + \sigma_1 B_t^{(1)} - \sigma_2 B_t^{(2)}\right).$$

Noting that $W^{(1)}, W^{(2)}$ have correlation coefficient ρ because $B^{(1)}, B^{(2)}$ have correlation coefficient ρ. Hence

$$\sigma_2 W_t^{(2)} - \sigma_1 W_t^{(1)}$$

is probabilistically equivalent to the process σW_t, where W_t is a standard Brownian motion and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{5.2}$$

We obtain

$$\sigma_1 B_t^{(1)} - \sigma_2 B_t^{(2)} = \int_0^t (t-s)^\alpha d(\sigma_1 W_s^{(1)} - \sigma_2 W_s^{(2)})$$

$$= -\sigma \int_0^t (t-s)^\alpha dW_s =: -\sigma B_t$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp(\mu t - \sigma B_t) \ , \ \ \mu = \mu_1 - \mu_2 \, . \tag{5.3}$$

We now can study the lifetime τ of the bank or of the insurance company, naturally define as the first value of t such that $X_t < Y_t$, or equivalently

$$\tau = \inf\{t : \ln \frac{X_t}{Y_t} < 0\} \, .$$

and the ruin probability on a finite time horizon $[0, t]$

$$\varphi(X_0, Y_0, t) := P(\tau < t) = P(\ln \frac{X_s}{Y_s} < 0 \text{ for some } s < t)$$

on an infinite time horizon

$$\varphi(X_0, Y_0) := \lim_{t\to\infty} \varphi(X_0, Y_0, t)$$

Using relation (5.3) we obtain

$$\varphi(X_0, Y_0) = P(\ln \frac{X_t}{Y_t} < 0 \text{ for some } t \geq 0)$$

$$= P(-\mu t + \sigma B_t > u \text{ for some } t \geq 0)$$

$$= P\big(\sup_{t\geq 0}(-\mu t + \sigma B_t) > u\big)$$

where $u = \ln \frac{X_0}{Y_0}$. In order to estimate $\varphi(X_0, Y_0)$ we use a result of Dębicki [5, Corollary 4.1] say that

Proposition 5.1. *For* $\frac{1}{2} \leq H \leq 1$

$$\lim_{u\to\infty} \frac{1}{u^{2-2H}} \ln P\big(A(B_H^o, c) > u\big) = -h \tag{5.4}$$

where $B_H^o(t) = \sqrt{2H} B_t$, $A(B_H^o, c) = \sup\{B_H^o(t) - ct : t \geq 0\}$ *and*

$$h = \frac{1}{2}\Big(\frac{c}{H}\Big)^{2H}\Big(\frac{1}{1-H}\Big)^{2-2H}$$

Now we can state the following theorem

Theorem 5.1. *If* $\mu_1 \geq \mu_2$ *then the ruin probability for ALM model (5.1) satisfy*

$$\lim_{u\to\infty} \frac{\ln \varphi(X_0, Y_0)}{u^{2-2H}} = -\frac{\mu^{2H}}{H\sigma^2}\Big(\frac{H}{1-H}\Big)^{2-2H}. \tag{5.5}$$

where $\mu = \mu_1 - \mu_2$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ *and* $u = \ln \frac{X_0}{Y_0}$.

Proof. We have

$$\varphi(X_0, Y_0) = P\Big(\sup_{t\geq 0}(B_t - \frac{\mu}{\sigma}t) > \frac{u}{\sigma}\Big)$$

$$= P\left(\sup_{t\geq 0}\Big(B_H^o(t) - \frac{\sqrt{2H}\mu}{\sigma}t\Big) > \frac{\sqrt{2H}u}{\sigma}\right)$$

therefore (5.5) follows from Proposition 5.1. The theorem is completed. □

Acknowledgment. The authors would like to thank for the anonymous referees for their valuable comments for improving the paper.

References

[1] E.Alòs, O.Mazet and D.Nualart. *Stochastic calculus with respect to fractional Brownian motion with Hurst paramenter less than* $\frac{1}{2}$. Stochastic processes and their applications **86** issue 1 (2000), 121-139.

[2] T. O. Androshchuk. *Approximation of a stochastic integral with respect to fractional Brownian motion by integrals with respect to absolutely continuous processes.* Theor. Probability and Math. Statist.No. 73, 2006, Pages 1929.

[3] F. Comte and E. Renault. *Long Memory in Continuous-Time Stochastic Volatility Models.* Mathematical Finance, Vol. 8, No. 4 (October 1998), 291-323.

[4] T.E.Duncan, Y.Hu and B.Pasik Duncan. *Stochastic Calculus for Fractional Brownian Motion.* SIAM Control and Optimization **38**, No 2 (2000), 582-612.

[5] K. Dębicki, Z. Michna, and T. Rolski, *On the supremum from Gaussian processes over infinite horizon.* Probability and Math. Stat. 18 (1998), 83100.

[6] N.T.Dung. *The European option pricing in exponential Lévy models.* Vietnam Journal of Mathematical Application , Vol. 5, No. 1 (June 2007), 1-12.

[7] D.Feyel and A.de la Pradelle. *Fractional integrals and Brownian processes.* Potential Analysis **10** (1996) 273-288.

[8] Y. Hu and B. Øksendal. *Fractional white noise calculus and applications to finance.* Preprint, Department of Mathematics, University of Oslo, 10, 1999.

[9] B.Mandelbrot, J.van Ness. *Fractional Brownian motions, Fractional Noises and Applications.* J. SIAM Review **10**, No. 4 (1968), 422-437.

[10] D.Nualart and A.Rascanu. *Differential equations driven by fractional Brownian motion.* Collectanea Mathematica **53** (2002) 55-81.

[11] Tran Hung Thao. *An approximate approach to fractional analysis for finance,* Nonlinear Analysis 7(2006),124-132.(available also online on Science Direct)

[12] Tran Hung Thao and Christine Thomas-Agnan. *Evolution des cours gouvernée par un processus de type ARIMA fractionnaire,* Studia Univ. Babes-Bolyai, Mathematica, Vol XVIII, No 2, June 2003.

[13] Thao, T.H and Nguyen , T.T (2003). Fractal Langevin Equation, Vietnam Journal of mathematics, 30:1, 89-96.

[14] M. Zähle, *Integration with respect to fractal functions and stochastic calculus.* Part I, Probab. Theory Related Fields 111 (1998), 33372.

[15] M. Zähle, *On the link between fractional and stochastic calculus.* Preprint of Jena University, Germany 1998.

Hanoi University of Natural Sciences, Viet Nam.

E-mail address: dung_nguyentien10@yahoo.com

Institute of Mathematics, Vietnam Academy of Science and Technology.