SEMIMARTINGALE APPROXIMATION OF FRACTIONAL BROWNIAN MOTION AND ITS APPLICATIONS

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ABSTRACT. The aim of this paper is to provide a semimartingale approximation of a fractional stochastic integration. This result leads us to approximate the fractional Black-Scholes model by a model driven by semimartingales, and a European option pricing formula is found.
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1. INTRODUCTION

The fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a centered Gaussian process defined by

$$W_t^{H,(1)} = \int_0^t K_1(t,s)dW_s\,, \tag{1.1}$$

where W is a standard Brownian motion and the kernel $K_1(t,s), t \geq s$, is given by

$$K_1(t,s) = C_H\left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})\int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}du\right],$$

where C_H is a coefficient depending only on H.

Another form of fractional Brownian motion is Liouville fractional Brownian motion (LfBm) [2, 6], where the kernel $K_1(t,s)$ is replaced by $K_2(t,s) = (t-s)^{H-\frac{1}{2}}$, that is a stochastic process defined by

$$W_t^{H,(2)} := \int_0^t (t-s)^{\alpha}dW_s\,,\ \alpha = H - \frac{1}{2}.$$

In [8] Mandelbrot has given a relation between $W_t^{H,(1)}$ and $W_t^{H,(2)}$

$$W_t^{H,(1)} = \frac{1}{\Gamma(1+\alpha)}\left[U_t + W_t^{H,(2)}\right], \tag{1.2}$$

where $U_t = \int\limits_{-\infty}^{0} \big((t-s)^\alpha - (-s)^\alpha\big) dW_s$ is a process of absolutely continuous trajectories.

It is well known that in the case where the Hurst index $H = \frac{1}{2}$, the process W^H ($W^H = W_t^{H,(1)}$ or $W_t^{H,(2)}$) is a standard Brownian motion and where $H \neq \frac{1}{2}$, W^H is neither a semimartingale nor a Markov process. Hence, the stochastic calculus developed by Itô cannot be applied. In this paper we use the pathwise stochastic integration, which is introduced by Zähle [16], to consider the following fractional version of the Black-Scholes (FB-S) model:

Bond price:

$$dB_t = rB_t dt; \quad B_0 = 1 \tag{1.3}$$

Stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^H, \tag{1.4}$$

where S_0 is a positive real number and W_t^H is either a fBm or a LfBm. The coefficients r, μ, σ are assumed to be constants symbolizing the riskless interest rate, the drift of the stock and its volatility, respectively.

The arbitrage in the (FB-S) model based on pathwise integration was studied by Shiryayev [12] for the case of $H > \frac{1}{2}$. Cheridito [3] proved a surprising result that, for Hurst parameters $H \in (\frac{3}{4}, 1)$ the mixed process $M_t^{H,\varepsilon} = W_t^{H,(1)} + \varepsilon W_t^1$ is equivalent to a martingale εW_t^1, as long as the standard Brownian motion W_t^1 is independent of $W_t^{H,(1)}$. He observes that

$$Cov(M_t^{H,\varepsilon}, M_s^{H,\varepsilon}) = \varepsilon^2 \min(t,s) + Cov(W_t^{H,(1)}, W_s^{H,(1)}).$$

Hence, $M_t^{H,\varepsilon}$ is an a.s. continuous centered Gaussian process that has up to ε^2 the same covariance structure as $(W_t^{H,(1)})$. Cheridito [3] verbally explains how this fact shows that if the stock price process in (FB-S) model fits empirical data, then so does

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{H,(1)} + \varepsilon\sigma S_t dW_t^1; \quad S_0 > 0 \tag{1.5}$$

for $\varepsilon > 0$ small enough.

It is obvious that mixed model (1.5) is arbitrage-free and complete. For a fixed value ε, one can price asset with respect to the unique martingale measure Q_ε and get at time $t = 0$

$$\begin{aligned} C_0(\varepsilon) &= E_{Q_\varepsilon}\left[\left(S_0 \exp(\mu T + \sigma(W_T^{H,(1)} + \varepsilon W_T^1)) - e^{-rT} K\right)^+\right] \\ &= BS(0, S_0, \sigma\varepsilon), \end{aligned}$$

where $BS(0, S_0, \sigma\varepsilon)$ denotes the Black-Scholes price of a call option on a stock with initial price S_0 and volatility $\sigma\varepsilon$. As $\varepsilon \to 0$, the mixed model (1.5) approaches the model (1.4), and the option price tends to

$$C_0 = \lim_{\varepsilon \to 0} BS(0, S_0, \sigma\varepsilon) = (S_0 - e^{-rT} K)^+, \tag{1.6}$$

that is, all randomness is eliminated. Cheridito [3] explains this peculiarity by the possibility that traders can act arbitrarily fast and hence immediately exploit the predictability of the model (1.5). Thereby, they remove the random character by means of a suitable trading strategy.

However, we can see that the mixed model (1.5) contains one random source more than the original model (1.4). This means that the dynamism of (1.5) is different from that of (1.4) even for arbitrarily small ε.

In [13, 14], T. H. Thao has proved that a LfBm can be approximated in $L^2(\Omega)$ by semimartingales. We developed this result by showing that W_t^H can be approximated in $L^p(\Omega)$ by semimartingales

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s) dW_s \ , \ \ \varepsilon > 0,$$

where $K(t, s)$ equals to either $K_1(t, s)$ or $K_2(t, s)$. This fact leads us to the following approximation model for stock price process

$$dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dW_t^{H,\varepsilon}; \quad S_0 > 0. \tag{1.7}$$

This model driven by semimartingales has the same random source as original (FB-S) model. We want also to emphasize that our approximation results is true for all $H > \frac{1}{2}$.

This paper is organized as follows: In Section 2, we state some basic facts about a semimartingale approximation of fractional processes and the generalized Stieltjes integral. In Section 3, our key result is stated in Theorem 3.1 that the fractional stochastic integral can be approximated by the stochastic integration with respect to semimartingales.

In Section 4, the absence of arbitrage and semimartingale approximation of the Black-Scholes model are proved, the Black-Scholes equation is found as well.

2. Preliminaries

Let us at first define the following stochastic process for every $\varepsilon > 0$

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s) dW_s \,,$$

where $K(t,s)$ equals to either $K_1(t,s)$ or $K_2(t,s)$. We have the following Proposition:

Proposition 2.1. I. *For every $\varepsilon > 0$, $W_t^{H,\varepsilon}$ is $\mathcal{F}_t$-semimartingale with following decomposition*

$$W_t^{H,\varepsilon} = \int_0^t K(s+\varepsilon, s) dW_s + \int_0^t \varphi_s^\varepsilon ds, \tag{2.1}$$

where $(\mathcal{F}_t, 0 \le t \le T)$ is the natural filtration associated to W.

$$\varphi_s^\varepsilon = \int_0^s \partial_1 K(s+\varepsilon, u) dW_u \,,$$

$$\partial_1 K(t,s) = \frac{\partial K(t,s)}{\partial t}.$$

II. *The process $W_t^{H,\varepsilon}$ converges to W_t^H in $L^p(\Omega), p > 0$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

Proof. The proof of part **I** is as follows: applying stochastic Fubini's theorem we have

(2.2)

$$\int_0^t \varphi_s^\varepsilon ds = \int_0^t \int_0^s \partial_1 K(s+\varepsilon, u) dW_u ds = \int_0^t \int_u^t \partial_1 K(s+\varepsilon, u) ds dW_u$$

$$= \int_0^t \big(K(t+\varepsilon, u) - K(u+\varepsilon, u)\big) dW_u = W_t^{H,\varepsilon} - \int_0^t K(s+\varepsilon, s) dW_s.$$

Hence, (2.1) follows from (2.2).

We are now in position to prove part **II** of the proposition. For any $p>0$, applying Burkholder-Davis-Gundy inequality (see, [10]) we get

$$E|W_t^{H,\varepsilon}-W_t^H|^p \le E\Big|\int_0^t \big(K(t+\varepsilon,s)-K(t,s)\big)dW_s\Big|^p \le c_p\Big(\int_0^t \big(K(t+\varepsilon,s)-K(t,s)\big)^2 ds\Big)^{\frac{p}{2}}, \tag{2.3}$$

where c_p is a finite positive constant and

$$\begin{aligned}\int_0^t \big(K(t+\varepsilon,s)-K(t,s)\big)^2 ds &= \int_0^t K^2(t+\varepsilon,s)ds - 2\int_0^t K(t+\varepsilon,s)K(t,s)ds + \int_0^t K^2(t,s)ds \\ &\le \int_0^{t+\varepsilon} K^2(t+\varepsilon,s)ds - 2\int_0^{t\wedge(t+\varepsilon)} K(t+\varepsilon,s)K(t,s)ds \\ &\quad + \int_0^t K^2(t,s)ds = E|W_{t+\varepsilon}^H - W_t^H|^2 \le \varepsilon^{2H}.\end{aligned} \tag{2.4}$$

Hence,

$$E|W_t^{H,\varepsilon}-W_t^H|^p \le c_p\varepsilon^{pH}.$$

The proof of the proposition is complete. $\square$

Corollary 2.1. *Let S_t^ε, S_t be the solution to equation (1.4), (1.7), respectively. Then S_t^ε converges to S_t in $L^p(\Omega), p>0$ when $\varepsilon\to 0$, provided that $H>\frac{1}{2}$. This convergence is uniform with respect to $t\in[0,T]$.*

Proof. Let X_1, X_2 be two random variables. By Lagrange's theorem and Hölder's inequality we have

$$\begin{aligned}E|e^{X_1}-e^{X_2}|^p &\le E\Big|(X_1-X_2)\sup_{\min(X_1,X_2)\le x\le\max(X_1,X_2)} e^x\Big|^p \\ &\le E|(X_1-X_2)e^{|X_1|+|X_2|}|^p \le \Big(E[e^{2p|X_1|}+e^{2p|X_2|}]E|X_1-X_2|^{2p}\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.5}$$

We recall from [4] that

$$S_t = S_0 e^{\mu t + \sigma W_t^H} \quad , \quad S_t^\varepsilon = S_0 e^{\mu t - \frac{1}{2}\sigma^2 K^2(t+\varepsilon,t) + \sigma W_t^{H,\varepsilon}}.$$

We now apply (2.5) to $X_1 = -\frac{1}{2}\sigma^2 K^2(t+\varepsilon,t) + \sigma W_t^{H,\varepsilon}, X_2 = \sigma W_t^H$ and obtain

$$\begin{aligned} E|S_t^\varepsilon - S_t|^p &= S_0^p e^{p\mu t} E|e^{-\frac{1}{2}\sigma^2 K^2(t+\varepsilon,t)+\sigma W_t^{H,\varepsilon}} - e^{\sigma W_t^H}|^p \\ &\leq S_0^p e^{p\mu t}\Big(E[e^{2p|X_1|} + e^{2p|X_2|}]E|X_1 - X_2|^{2p}\Big)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

It is obvious that $E[e^{2p|X_1|} + e^{2p|X_2|}]$ is finite because $W_t^{H,\varepsilon}$ and W_t^H are centered Gaussian processes with finite variances in $[0,T]$. Moreover, by fundamental inequality $(a+b)^p \leq c_p(a^p + b^p)$, where $c_p = 1$ if $0 < p \leq 1$ and $c_p = 2^{p-1}$ if $p > 1$

$$\begin{aligned} E|X_1 - X_2|^{2p} &\leq c_{2p}\big[E|W_t^{H,\varepsilon} - W_t^H|^{2p} + \frac{1}{4^p}\sigma^{4p}K^{4p}(t+\varepsilon,t)\big] \\ &\leq c_{2p}(\varepsilon^{2pH} + \frac{1}{4^p}\sigma^{4p}\varepsilon^{4p(H-\frac{1}{2})}). \end{aligned} \tag{2.7}$$

Thus, for $0 < \varepsilon < 1$, there exists a finite constant $C(p, S_0, T)$ depending only on p, S_0 and T such that

$$E|S_t^\varepsilon - S_t|^p \leq C(p, S_0, T)\varepsilon^{2p(H-\frac{1}{2})}. \tag{2.8}$$

□

Next, we recall about a generalization of the Stieltjes integral introduced by Zähle [16]. Fix a parameter $0 < \lambda < \frac{1}{2}$, denote by $W^{1-\lambda,\infty}[0,T]$ the space of measurable function $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\lambda,\infty} := \sup_{0\leq s<t\leq T}\Big(\frac{|g(t)-g(s)|}{(t-s)^{1-\lambda}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\lambda}}dy\Big) < +\infty\,.$$

Clearly,

$$C^{1-\lambda+\varepsilon}[0,T] \subset W^{1-\lambda,\infty}[0,T] \subset C^{1-\lambda}[0,T] \;\; \forall \;\; \varepsilon > 0,$$

where $C^\lambda[0,T]$ denotes the space of Hölder continuous functions of order λ with the norm

$$\|g\|_\lambda := \sup_{0\leq t\leq T}|g(t)| + \sup_{0\leq s<t\leq T}\frac{|g(t)-g(s)|}{|t-s|^\lambda}\,.$$

We also denote by $W^{\lambda,1}[0,T]$ the space of measurable function $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\lambda,1} := \int_0^T \frac{|f(s)|}{s^\lambda} ds + \int_0^T \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\lambda+1}} dsdt < \infty .$$

For the functions $f \in W^{\lambda,1}[0,T], g \in W^{1-\lambda,\infty}[0,T]$, Zähle introduced the generalized Stieltjes integral

$$\int_0^T f(t)dg(t) = (-1)^\lambda \int_0^T D^\lambda_{0+} f(t) D^{1-\lambda}_{T^-} g(t)dt$$

defined in terms of the fractional derivative operators

$$D^\lambda_{0+} f(x) = \frac{1}{\Gamma(1-\lambda)} \left(\frac{f(x)}{x^\lambda} + \lambda \int_0^x \frac{f(x)-f(y)}{(x-y)^{\lambda+1}} dy \right),$$

and

$$D^{1-\lambda}_{T^-} g(x) = \frac{(-1)^\lambda}{\Gamma(1-\lambda)} \left(\frac{g(x)-g(T)}{(T-x)^\lambda} + \lambda \int_x^T \frac{g(x)-g(y)}{(x-y)^{\lambda+1}} dy \right).$$

Moreover, we have the following estimate for all $t \in [0,T]$

$$|\int_0^t f dg| \le C(\lambda) \|f\|_{\lambda,1} \|g\|_{1-\lambda,\infty} . \tag{2.9}$$

If $f \in C^\lambda[0,T]$ and $g \in C^\mu[0,T]$ with $\lambda+\mu > 1$, it is proved by Zähle that the integral $\int_0^t f dg$ coincides with the Riemann-Stieltjes integral.

3. Approximation results

Theorem 3.1. *Suppose that u_t is a stochastic process belonging to $C^{1-H+\delta}[0,T]$ a.s. with some constant $\delta > 0$, i.e.*

$$\sup_{0\le t\le T} |u_t| + \sup_{0\le s<t\le T} \frac{|u_t-u_s|}{|t-s|^{1-H+\delta}} \le K^2(\omega) \ a.s. \tag{3.1}$$

where $K(\omega)$ is a finite random variable. Then

$$\int_0^T u_s dW_s^{H,\varepsilon} \xrightarrow{P} \int_0^T u_s dW_s^H \quad \textit{when } \varepsilon \to 0, \tag{3.2}$$

provided that $H > \frac{1}{2}$. The notation $\xrightarrow{P}$ stands for the convergence in probability.

Proof. For every $\varepsilon > 0$ we consider

$$u_t^\varepsilon = \sum_{i=1}^n u_{t_{i-1}} \mathbf{1}_{[t_{i-1},t_i)}(t) \;\;,\;\; u_T^\varepsilon = u_T,$$

where $n = [\frac{T}{\varepsilon} + 1]\,, t_i = \frac{iT}{n}\,, i = 0, ..., n$ and $\mathbf{1}$ is the indicator function, i.e. $\mathbf{1}_{[t_{i-1},t_i)}(t) = \begin{cases} 1 & \text{if } t \in [t_{i-1}, t_i) \\ 0 & \text{otherwise.} \end{cases}$

For any $t \in [0, T], t$ should belong to some interval $[t_{i-1}, t_i)$ for some i, then the condition (3.1) leads us to the following estimate

$$\begin{aligned} |u_t^\varepsilon - u_t| = |u_t - u_{t_{i-1}}| &\leq K^2(\omega)|t - t_{i-1}|^{1-H+\delta} \\ &\leq K^2(\omega)|t_i - t_{i-1}|^{1-H+\delta} \leq K^2(\omega)\varepsilon^{1-H+\delta} \;\; a.s. \end{aligned} \tag{3.3}$$

It is easy to see that

$$\begin{aligned} \left| \int_0^T u_s dW_s^{H,\varepsilon} - \int_0^T u_s dW_s^H \right| &\leq \left| \int_0^T (u_s^\varepsilon - u_s) dW_s^H \right| \\ &+ \left| \int_0^T (u_s^\varepsilon - u_s) dW_s^{H,\varepsilon} \right| + \left| \int_0^T u_s^\varepsilon d(W_s^{H,\varepsilon} - W_s^H) \right|. \end{aligned} \tag{3.4}$$

Firstly, we prove that the first term in the right-hand side of (3.4) converges to 0 in probability. Fix a parameter $1 - H < \lambda < \min\{\frac{1}{2}, 1 - H + \delta\}$, applying the inequality (2.9) we have

$$\left| \int_0^T (u_s^\varepsilon - u_s) dW_s^H \right| \leq C(\lambda) \left\| u^\varepsilon - u \right\|_{\lambda,1} \left\| W^H \right\|_{1-\lambda,\infty} \text{ a.s.,} \tag{3.5}$$

where $C(\lambda)$ is a finite positive constant and

$$\|u^\varepsilon - u\|_{\lambda,1} = \int_0^T \frac{|u_s^\varepsilon - u_s|}{s^\lambda} ds + \int_0^T \int_0^t \frac{|u_t^\varepsilon - u_t - u_s^\varepsilon + u_s|}{(t-s)^{\lambda+1}} dsdt$$
$$\leq \frac{T^{1-\lambda}}{1-\lambda} \sup_{0\leq s\leq T} |u_s^\varepsilon - u_s| + \int_0^T \int_0^t \frac{|u_t^\varepsilon - u_t - u_s^\varepsilon + u_s|}{(t-s)^{\lambda+1}} dsdt$$
$$\leq \frac{T^{1-\lambda}}{1-\lambda} K^2(\omega)\varepsilon^{1-H+\delta} + \int_0^T \int_0^t \frac{|u_t^\varepsilon - u_t - u_s^\varepsilon + u_s|}{(t-s)^{\lambda+1}} dsdt.$$

Noting that for every fixed $t \in [0,T]$ there exists $\varepsilon > 0$ such that $t \in [t_{i-1}, t_i)$ with some i. We have

$$\int_0^t \frac{|u_t^\varepsilon - u_t - u_s^\varepsilon + u_s|}{(t-s)^{\lambda+1}} ds = \sum_{k=1}^{i-1} \int_{t_{k-1}}^{t_k} \frac{|u_{t_{i-1}} - u_t - u_{t_{k-1}} + u_s|}{(t-s)^{\lambda+1}} ds \tag{3.6}$$
$$+ \int_{t_{i-1}}^t \frac{|u_{t_{i-1}} - u_t - u_{t_{i-1}} + u_s|}{(t-s)^{\lambda+1}} ds \leq \sum_{k=1}^{i-1} \int_{t_{k-1}}^{t_k} \frac{2K^2(\omega)\varepsilon^{1-H+\delta}}{(t-s)^{\lambda+1}} ds$$
$$+ \int_{t_{i-1}}^t \frac{K^2(\omega)|t-s|^{1-H+\delta}}{(t-s)^{\lambda+1}} ds = \frac{2K^2(\omega)\varepsilon^{1-H+\delta}}{\lambda} [(t-t_{i-1})^{-\lambda} - t^{-\lambda}]$$
$$+ \frac{K^2(\omega)}{1-H-\lambda+\delta} (t-t_{i-1})^{1-H-\lambda+\delta}$$
$$\leq \frac{2K^2(\omega)\varepsilon^{1-H+\delta}}{\lambda} (t-t_{i-1})^{-\lambda} + \frac{K^2(\omega)\varepsilon^{1-H-\lambda+\delta}}{1-H-\lambda+\delta}.$$

Hence,

$$\|u^\varepsilon - u\|_{\lambda,1} \leq \frac{K^2(\omega)}{1-H-\lambda+\delta} (t-t_{i-1})^{1-H-\lambda+\delta} \tag{3.7}$$
$$+ \frac{2K^2(\omega)\varepsilon^{1-H+\delta}}{\lambda} \int_0^T (t-t_{i-1})^{-\lambda} dt + \frac{K^2(\omega)\varepsilon^{1-H-\lambda+\delta}}{1-H-\lambda+\delta} \to 0$$

as $\varepsilon \to 0$ because the integral in the right-hand side of (3.7) is finite.

It is well known that W^H has $(H-\eta)$-Hölder continuous paths for all $\eta \in (0,H)$ (see, [8]), i.e. there exists a finite random variable $K_\eta(\omega)$

such that

$$|W_t^H - W_s^H| \leq K_\eta(\omega)|t-s|^{H-\eta} \ \forall\, t, s \in [0,T] \text{ a.s.}$$

For $0 < \eta < \lambda - (1-H)$ we have

$$\|W^H\|_{1-\lambda,\infty} = \sup_{0\leq s<t\leq T} \Big(\frac{|W_t^H - W_s^H|}{(t-s)^{1-\lambda}} + \int_s^t \frac{|W_y^H - W_s^H|}{(y-s)^{2-\lambda}} dy \Big) \tag{3.8}$$

$$\leq K_\eta(\omega) \sup_{0\leq s<t\leq T} \Big((t-s)^{H+\lambda-\eta-1} + \int_s^t (y-s)^{H+\lambda-\eta-2} dy \Big)$$

$$\leq K_\eta(\omega) T^{H+\lambda-\eta-1} \Big(1 + \frac{1}{H+\lambda-\eta-1} \Big).$$

As a consequence, by combining (3.5), (3.7) and (3.8) the first term in the right-hand side of (3.4) will converge to zero in probability as $\varepsilon \to 0$.

Next, we prove the second term in the right-hand side of (3.4) converges to zero in $L^2(\Omega)$ by using the decomposition (2.1).

$$\begin{aligned} E\Big|\int_0^T (u_s^\varepsilon - u_s) dW_s^{H,\varepsilon}\Big|^2 &\leq E\Big|\int_0^T (u_s^\varepsilon - u_s) K(s+\varepsilon, s) dW_s\Big|^2 \\ &+ E\Big|\int_0^T (u_s^\varepsilon - u_s)\varphi_s^\varepsilon ds\Big|^2 \leq \int_0^T E(u_s^\varepsilon - u_s)^2 K^2(s+\varepsilon, s) ds \\ &+ \varepsilon^{2-2H+2\delta} \int_0^T E\big|K^2(\omega)\varphi_s^\varepsilon\big|^2 ds. \end{aligned} \tag{3.9}$$

It is obvious that the first term in the right-hand side of (3.9) converges to zero in $L^2(\Omega)$ because $E(u_s^\varepsilon - u_s)^2 \leq E[K^4(\omega)]\varepsilon^{2-2H+2\delta}$ and $K(s+\varepsilon, s) \to K(s,s) = 0$ as $\varepsilon \to 0$.

Applying the Hölder and the Burkholder-Davis-Gundy inequalities we have

$$E\big|K^2(\omega)\varphi_s^\varepsilon\big|^2 \le \big(E|K(\omega)|^8\big)^{1/2}\big(E|\int_0^s \partial_1 K(s+\varepsilon,u)dW_u|^4\big)^{1/2}$$
$$\le C\int_0^s |\partial_1 K(s+\varepsilon,u)|^2 du,$$

where C is a finite constant. We recall that

$$\partial_1 K(t,s) = \begin{cases} C_H \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{3}{2}} & \text{if } W_t^H = W_t^{H,(1)}, \\ (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} & \text{if } W_t^H = W_t^{H,(2)}. \end{cases}$$

There exists C' not depending on ε such that

$$E\big|K^2(\omega)\varphi_s^\varepsilon\big|^2 \le C'\int_0^s (s+\varepsilon-u)^{2H-3}du = \frac{C'}{2-2H}[\varepsilon^{2H-2}-(s+\varepsilon)^{2H-2}],$$

and so the second term in the right-hand side of (3.9) converges to zero in $L^2(\Omega)$.

Finally, we prove that the third term in the right-hand side of (3.4) converges to 0.

$$\int_0^T u_s^\varepsilon d(W_s^{H,\varepsilon}-W_s^H) \tag{3.10}$$
$$= \sum_{i=1}^n u_{t_{i-1}}(W_{t_i}^{H,\varepsilon}-W_{t_i}^H-W_{t_{i-1}}^{H,\varepsilon}+W_{t_{i-1}}^H)+u_T(W_T^{H,\varepsilon}-W_T^H)$$
$$= \sum_{i=1}^n (u_{t_{i-1}}-u_{t_i})(W_{t_i}^{H,\varepsilon}-W_{t_i}^H)+u_T(W_T^{H,\varepsilon}-W_T^H).$$

It is obvious that $u_T(W_T^{H,\varepsilon}-W_T^H)\xrightarrow{L^2(\Omega)}0$ because $W_T^{H,\varepsilon}\xrightarrow{L^2(\Omega)}W_T^H$. Moreover, we have

$$|\sum_{i=1}^n (u_{t_{i-1}}-u_{t_i})(W_{t_i}^{H,\varepsilon}-W_{t_i}^H)| \tag{3.11}$$
$$\le K^2(\omega)\sum_{i=1}^n |t_{i-1}-t_i|^{1-H+\delta}|W_{t_i}^{H,\varepsilon}-W_{t_i}^H|,$$

and

$$E\sum_{i=1}^{n}|t_{i-1}-t_i|^{1-H+\delta}|W^{H,\varepsilon}_{t_i}-W^H_{t_i}| \tag{3.12}$$
$$\leq \sum_{i=1}^{n}|t_{i-1}-t_i|^{1-H+\delta}\Big(E|W^{H,\varepsilon}_{t_i}-W^H_{t_i}|^2\Big)^{\frac{1}{2}}$$
$$\leq \sum_{i=1}^{n}\big(\frac{T}{n}\big)^{1-H+\delta}\varepsilon^H \leq \sum_{i=1}^{n}\varepsilon^{1-H+\delta}\varepsilon^H$$
$$= [\frac{T}{\varepsilon}+1]\varepsilon^{1+\delta} \to 0 \text{ when } \ \varepsilon \to 0.$$

Thus, the proof of the theorem is complete. □

Remark 3.1. Another approximation approach is given by Androshchuk [1] who proved that for a stochastic process $u \in C^{2-2H+\delta}[0,T] \subset C^{1-H+\delta}[0,T]$ $a.s$ the fractional stochastic integral can be approximated by integrals with respect to absolutely continuous processes. More applications to finance is introduced by Mishura [9].

4. Applications to Fractional Black-Scholes model

Theorem 4.1. *Suppose that $H \in (0,1)$. For fixed $\varepsilon > 0$, the approximation model (1.3) and (1.7) has no arbitrage.*

Proof. Using (2.1) we can rewrite (1.7) as follows

$$dS^\varepsilon_t = (\mu + \sigma\varphi^\varepsilon_s)S^\varepsilon_t dt + \sigma K(t+\varepsilon,t)S^\varepsilon_t dW_t; \quad S_0 > 0. \tag{4.1}$$

From [11, Theorem 12.1.8] we have only to prove that the stochastic process

$$u(t,\omega) := \frac{\mu+\sigma\varphi^\varepsilon_t - r}{\sigma K(t+\varepsilon,t)}$$

satisfies the Novikov's condition

$$E\Big[\exp\big(\frac{1}{2}\int\limits_0^T u^2(t,\omega)dt\big)\Big] < \infty.$$

The latest inequality holds obviously because $\varphi^\varepsilon_t = \int\limits_0^t \partial_1 K(t+\varepsilon,u)dW_u$ is a Gaussian process with finite variance.

The proof of Theorem thus is complete. □

A strategy in this model is a pair of adapted stochastic processes $\pi = (\alpha_t, \beta_t)$, where the processes α_t and β_t denote the number of bonds at time t and number of stock shares held at time t, respectively. Thus, the corresponding wealth process is given by

$$V_t = \alpha_t B_t + \beta_t S_t,$$

where B_t and S_t are the bond price and stock price at time t, respectively.

We make the following assumptions about the strategy π :

(A_1). π is a self-financing strategy, i.e.

$$V_t = V_0 + \int_0^t \alpha_s dB_s + \int_0^t \beta_s dS_s.$$

where the second integral in the right-hand side is a pathwise integral. (A_2). π is a strategy of the following form (Markov-type strategy)

$$\alpha_t = \alpha(t, S_t) \quad , \quad \beta_t = \beta(t, S_t).$$

Next, we will prove that in the class of the Markov-type strategies the wealth process can be considered as a limit of semimartingales. Indeed, we have

$$V_t^\varepsilon = \alpha(t, S_t^\varepsilon) B_t + \beta(t, S_t^\varepsilon) S_t^\varepsilon$$

or equivalently,

$$\begin{aligned} V_t^\varepsilon &= V_0 + \int_0^t \alpha(s, S_s^\varepsilon) dB_s + \int_0^t \beta(s, S_s^\varepsilon) dS_s^\varepsilon \\ &= V_0 + \int_0^t \left[\alpha(s, S_s^\varepsilon) r B_s + \mu\beta(s, S_s^\varepsilon) S_s^\varepsilon\right] ds + \int_0^t \sigma\beta(s, S_s^\varepsilon) S_s^\varepsilon dW_s^{H,\varepsilon}. \end{aligned}$$

From the semimartingale decomposition (2.1) we obtain

$$\begin{aligned} V_t^\varepsilon = V_0 + \int_0^t \left[r\alpha(s, S_s^\varepsilon) B_s + \mu\beta(s, S_s^\varepsilon) S_s^\varepsilon + \sigma\varphi_s^\varepsilon \beta(s, S_s^\varepsilon) S_s^\varepsilon\right] ds \\ + \int_0^t \sigma K(s+\varepsilon, s) \beta(s, S_s^\varepsilon) S_s^\varepsilon dW_s \end{aligned} \tag{4.2}$$

which means that V_t^ε is a semimartingale.

Theorem 4.2. *Let $H > \frac{1}{2}$ and assume that the self-financing, Markov-type strategy π satisfies the following conditions with some constants $\delta_1, \delta_2, \delta_3 > 0$*

(C_1). $|\alpha(t,x) - \alpha(t,y)| \leq M|x-y|^{\delta_1}\ \forall\ x,y \in \mathbb{R}\ \forall\ t \in [0,T]$.

(C_2). $|\beta(t,x) - \beta(s,x)| \leq M|t-s|^{\frac{1}{2}+\delta_2}\ \forall\ x \in \mathbb{R}\ \forall\ t,s \in [0,T]$.

(C_3). *$\beta(t,x)$ is a differentiable function in x and*

$$|\beta'_x(t,x)| \leq M(1+|x|^{\delta_3})\ \forall\ x \in \mathbb{R}.$$

Then $V_t^\varepsilon \xrightarrow{P} V_t$ as $\varepsilon \to 0$ for any $t \in [0,T]$.

Proof. We have

$$\begin{aligned} V_t &= V_0 + \int_0^t \alpha(s,S_s)dB_s + \int_0^t \beta(s,S_s)dS_s \\ &= V_0 + \int_0^t \big[\alpha(s,S_s)rB_s + \mu\beta(s,S_s)S_s\big]ds + \int_0^t \sigma\beta(s,S_s)S_s dW_s^H, \end{aligned}$$

$$\begin{aligned} V_t^\varepsilon &= V_0 + \int_0^t \alpha(s,S_s^\varepsilon)dB_s + \int_0^t \beta(s,S_s^\varepsilon)dS_s^\varepsilon \\ &= V_0 + \int_0^t \big[\alpha(s,S_s^\varepsilon)rB_s + \mu\beta(s,S_s^\varepsilon)S_s^\varepsilon\big]ds + \int_0^t \sigma\beta(s,S_s^\varepsilon)S_s^\varepsilon dW_s^{H,\varepsilon}. \end{aligned}$$

Hence,

$$\begin{aligned} |V_t^\varepsilon - V_t| \leq& \int_0^t \big|\alpha(s,S_s^\varepsilon) - \alpha(s,S_s)\big| r\, e^{rs} ds \\ &+ \mu \int_0^t \big|\beta(s,S_s^\varepsilon)S_s^\varepsilon - \beta(s,S_s)S_s\big| ds \\ &+ \left| \int_0^t \sigma\beta(s,S_s^\varepsilon)S_s^\varepsilon dW_s^{H,\varepsilon} - \int_0^t \sigma\beta(s,S_s)S_s dW_s^H \right| := I_1 + I_2 + I_3. \end{aligned} \tag{4.3}$$

First, by using Hölder's inequality and the condition (C_1) we get

$$\begin{aligned}(4.4)\quad E[I_1^2] &\leq \int_0^t r^2\, e^{2rs} ds \int_0^t E|\alpha(s,S_s^\varepsilon)-\alpha(s,S_s)|^2 ds \\ &\leq \frac{rM^2(e^{2rT}-1)}{2}\int_0^t E|S_s^\varepsilon - S_s|^{2\delta_1} ds.\end{aligned}$$

Consequently, Corollary 2.1 implies that $I_1 \xrightarrow{L^2(\Omega)} 0$ when $\varepsilon \to 0$.

Next, we prove that I_2 also converges to 0 in $L^2(\Omega)$. Indeed, put $f(t,x) = \beta(t,x)x$, $u_t^\varepsilon = f(t,S_t^\varepsilon)$ and $u_t = f(t,S_t)$ then by Hölder's inequality we have

$$\begin{aligned}(4.5)\quad E|u_t^\varepsilon - u_t|^2 &\leq E\big[A(t,x)(S_t^\varepsilon - S_t)\big]^2 \\ &\leq \big[E|A(t,x)|^4\big]^{\frac12}\big[E|S_t^\varepsilon - S_t|^4\big]^{\frac12},\end{aligned}$$

where

$$A(t,x) := \sup_{\min(S_t^\varepsilon,S_t)\leq x\leq \max(S_t^\varepsilon,S_t)} \Big|\frac{\partial f(t,x)}{\partial x}\Big|.$$

From Corollary 2.1 we have

$$(4.6)\qquad \big[E|S_t^\varepsilon - S_t|^4\big]^{\frac12} \leq C(S_0,T)\varepsilon^{4H-2} \to 0$$

uniformly in $t\in[0,T]$ as $\varepsilon\to 0$. Therefore, we need only to prove the first term in the right-hand side of (4.5) is finite.

Using the conditions (C_2) and (C_3) we have

$$\begin{aligned}\Big|\frac{\partial f(t,x)}{\partial x}\Big| &\leq |\beta(t,x)| + |\beta_x'(t,x)x| \\ &\leq |\beta(0,x)| + Mt^{\frac12+\delta_2} + M(|x|+|x|^{1+\delta_3}).\end{aligned}$$

Hence,

$$\begin{aligned}A(t,x) &\leq \sup_{\min(S_t^\varepsilon,S_t)\leq x\leq \max(S_t^\varepsilon,S_t)} (|\beta(0,x)| + Mt^{\frac12+\delta_2} + M(|x|+|x|^{1+\delta_3})) \\ &\leq |\beta(0,S_0)| + MT^{\frac12+\delta_2} + M \sup_{\min(S_t^\varepsilon,S_t)\leq x\leq \max(S_t^\varepsilon,S_t)} (|x|+|x|^{1+\delta_3}) \\ &\leq |\beta(0,S_0)| + MT^{\frac12+\delta_2} + M \sup_{|x|\leq S_t^\varepsilon + S_t} (|x|+|x|^{1+\delta_3}) \\ &\leq |\beta(0,S_0)| + MT^{\frac12+\delta_2} + M(|S_t^\varepsilon + S_t| + |S_t^\varepsilon + S_t|^{1+\delta_3})\end{aligned}$$

and the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ leads us to

$$E|A(t,x)|^4 \leq 27\big[(\beta(0,S_0)+MT^{\frac{1}{2}+\delta_2})^4 + M^4E|S_t^\varepsilon + S_t|^4 \\ + M^4E|S_t^\varepsilon+S_t|^{4(1+\delta_3)}\big].$$

Now it is enough to prove $E|S_t^\varepsilon + S_t|^p < \infty$ for any $p > 1$. We have

$$(4.7)\quad E|S_t^\varepsilon + S_t|^p \leq 2^{p-1}(E|S_t^\varepsilon|^p + E|S_t|^p) \\ \leq 2^{p-1}S_0^p\big(Ee^{p(\mu t - \frac{1}{2}\sigma^2K(t+\varepsilon,t)+\sigma W_t^{H,\varepsilon})} + Ee^{p(\mu t+\sigma W_t^H)}\big).$$

Obviously, the right-hand side of (4.7) is bounded by a constant C_T because $W_t^{H,\varepsilon}, W_t^H, t \in [0,T]$ are centered Gaussian processes with finite variances. Thus, $u_t^\varepsilon \xrightarrow{L^2(\Omega)} u_t$ uniformly in $t \in [0,T]$ when $\varepsilon \to 0$ and

$$E|I_2|^2 = \mu^2 E\bigg(\int_0^t (u_s^\varepsilon - u_s)ds\bigg)^2 \leq \mu^2T^2 \sup_{0\leq s\leq T} E|u_s^\varepsilon - u_s|^2 \to 0\ ,\ \varepsilon \to 0.$$

Finally, we show that $I_3 \xrightarrow{P} 0$ when $\varepsilon \to 0$.

$$(4.8)\quad I_3 = \bigg|\int_0^t \sigma u_s^\varepsilon dW_s^{H,\varepsilon} - \int_0^t \sigma u_s dW_s^H\bigg| \\ \leq \bigg|\int_0^t \sigma(u_s^\varepsilon - u_s)dW_s^{H,\varepsilon}\bigg| + \bigg|\int_0^t \sigma u_s dW_s^{H,\varepsilon} - \int_0^t \sigma u_s dW_s^H\bigg|$$

Since $S_t \in C^{\frac{1}{2}^-}[0,T] = \bigcap\limits_{\delta<\frac{1}{2}} C^\delta[0,T]$ and under the conditions $(C_2), (C_3)$, the simple estimate

$$|u_t - u_s| \leq |\beta(t,S_t)S_t - \beta(s,S_t)S_t| + |\beta(s,S_t)S_t - \beta(s,S_s)S_s|$$

implies that $u_t \in C^{\frac{1}{2}^-}[0,T]$. Hence, the convergence of the second term in the right-hand side of (4.8) to zero in probability follows from Theorem 3.1. The first term converges to zero in probability because of Lemma 4.1 below. Indeed, we have from the chain rule for Malliavin derivative

$$D_su_t = D_s[\beta(t,S_t)S_t] = [\beta'_x(t,S_t)S_t + \beta(t,S_t)]D_s[S_0e^{\mu t+\sigma W_t^H}] \\ = \sigma S_t[\beta'_x(t,S_t)S_t + \beta(t,S_t)]K(t,s)$$

which implies that the condition (4.9) below holds.

The proof of Theorem thus is complete. □

Denote by $\mathbb{D}^{1,2} \subset L^2(\Omega)$ the space of Malliavin differentiable variables with norm

$$\|F\|_{1,2} := \big[E|F|^2\big]^{\frac{1}{2}} + E\Big[\int_0^T |D_u F|^2 du\Big]^{\frac{1}{2}}.$$

Lemma 4.1. *Suppose that $H > \frac{1}{2}$. Let $u, u^\varepsilon \in \mathbb{D}^{1,2}$ be adapted stochastic processes satisfying the condition*

$$\int_0^T \int_0^t |D_s u_t| \partial_1 K(t,s) ds dt < \infty \ \ a.s. \tag{4.9}$$

If $u_t^\varepsilon \to u_t$ ucp (uniform convergence in probability), that is $\forall\, t : |u_t^\varepsilon - u_t| \le C\varepsilon^\gamma$ a.s with some $\gamma > 0$ then

$$\lim_{\varepsilon\to 0} \int_0^T (u_s^\varepsilon - u_s) dW_s^{H,\varepsilon} = 0 \tag{4.10}$$

in probability.

Proof. From the decomposition (2.1) we have

$$\int_0^T (u_s^\varepsilon - u_s)\, dW_s^{H,\varepsilon} = \int_0^T (u_s^\varepsilon - u_s) K(s+\varepsilon, s) dW_s + \int_0^T (u_s^\varepsilon - u_s) \int_0^s \partial_1 K(s+\varepsilon, t)\, dW_t ds.$$

Since $\lim_{\varepsilon\to 0} \int_0^s \partial_1 K(s+\varepsilon, t)\, dW_t$ does not exist, we can not take the limit as $\varepsilon \to 0$ directly. However, the anticipating stochastic Fubini's theorem (see Theorem 3.1 [7]) yields

$$\int_0^T (u_s^\varepsilon - u_s)\, dW_s^{H,\varepsilon} = \int_0^T (u_s^\varepsilon - u_s) K(T+\varepsilon, s)\, dW_s$$
$$+ \int_0^T \int_s^T (u_t^\varepsilon - u_t - u_s^\varepsilon + u_s)\, \partial_1 K(s+\varepsilon, t) dt \delta W_s$$
$$+ \int_0^T dt \int_0^t D_s(u_t^\varepsilon - u_t)\, \partial_1 K(t+\varepsilon, s) ds := A_1 + A_2 + A_3,$$

where D_sF is the Malliavin derivative of variable F and δW_s is the Skorokhod differential.

It is easy to see that $A_1, A_2 \to 0$ because $u_t^\varepsilon \to u_t$ *ucp* and the condition (4.9) is enough to ensure the convergence of A_3 to zero.

The proof of Lemma thus is complete. □

Theorem 4.3. *Suppose that $H > \frac{1}{2}$. Let $C(t, S_t^\varepsilon)$ denote the value of a European call option at time t in the approximation (FB-S) model (1.3), (1.7). Then the Black-Scholes equation is given by*

$$\frac{1}{2}\sigma^2 K^2(t+\varepsilon, t)(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2} + r\frac{\partial C}{\partial S^\varepsilon} S^\varepsilon + \frac{\partial C}{\partial t} - rC = 0 \tag{4.11}$$

and as a consequence, the Black-Scholes equation in (FB-S) model is

$$r\frac{\partial C}{\partial S} S + \frac{\partial C}{\partial t} - rC = 0 \tag{4.12}$$

which gives us the explicit formula for price of a European call option at time $t = 0$

$$C_0 = (S_0 - e^{-rT} K)^+. \tag{4.13}$$

Proof. Using Itô's differential formula, we get

$$dC = [\frac{\partial C}{\partial t} + (\mu + \sigma \varphi_t^\varepsilon) S^\varepsilon \frac{\partial C}{\partial S^\varepsilon} + \frac{1}{2}\sigma^2 K^2(t+\varepsilon, t)(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2}] dt + \sigma K(t+\varepsilon, t) \frac{\partial C}{\partial S^\varepsilon} S^\varepsilon dW_t. \tag{4.14}$$

We form a portfolio consisting of

- one unit of the option C,

- a short position on $\frac{\partial C}{\partial S^\varepsilon}$ units of the stock S^ε and
- a debt of $A(t, S^\varepsilon)$ at the risk-free interest rate r.

The value process $R(t, S_t^\varepsilon)$ of this portfolio satisfies

$$dR = dC - \frac{\partial C}{\partial S^\varepsilon} dS^\varepsilon - Ar\,dt$$
$$= [\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 K^2(t+\varepsilon, t)(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2} - Ar]dt.$$

Now we choose

$$A = \frac{1}{r}[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 K^2(t+\varepsilon, t)(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2}]$$

then $dR = 0$. Obviously, the portfolio does not yield any return, hence its value itself must also be zero. This leads to the Black-Scholes partial differential equation

$$\frac{1}{2}\sigma^2 K^2(t+\varepsilon, t)(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2} + r\frac{\partial C}{\partial S^\varepsilon} S^\varepsilon + \frac{\partial C}{\partial t} - rC = 0 \tag{4.15}$$

which has to be solved with respect to the boundary conditions

$$\begin{cases} C(t, 0) = 0 \ \forall\ t \in [0, T], \\ C(T, S_T^\varepsilon) = (S_T^\varepsilon - K)^+. \end{cases}$$

The equation (4.12) follows from (4.15) by taking the limit as $\varepsilon \to 0$.

The proof of Theorem thus is complete. □

Remark 4.1. The equation (4.15) holds for all $H \in (0, 1)$ and in the case, $W_t^H = W_t^{H,(2)}$ is LfBm, it becomes

$$\frac{1}{2}\sigma^2 \varepsilon^{2\alpha}(S^\varepsilon)^2 \frac{\partial^2 C}{\partial (S^\varepsilon)^2} + r\frac{\partial C}{\partial S^\varepsilon} S^\varepsilon + \frac{\partial C}{\partial t} - rC = 0 \tag{4.16}$$

and we get the price of a European call option

$$C_0(\varepsilon) = S_0 N(d_1) - e^{-rT} K N(d_2)$$

where $d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{\sigma^2\varepsilon^{2\alpha}}{2})T}{\sigma\varepsilon^\alpha\sqrt{T}}$, $d_2 = \frac{\ln\frac{S_0}{K} + (r - \frac{\sigma^2\varepsilon^{2\alpha}}{2})T}{\sigma\varepsilon^\alpha\sqrt{T}}$ and $N(x)$ is the standard normal cumulative distribution function.

Obviously, for $H = \frac{1}{2}$, we get the well-known Black-Scholes pricing formula.

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