Mackey-Glass equation driven by fractional Brownian motion

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Abstract

In this paper we introduce a fractional stochastic version of the Mackey-Glass model which is a potential candidate to model objects in biology and finance. By a semimartingale approximate approach we find a semi-analytical expression for the solution.

Keywords: Mackey-Glass equation, Fractional Brownian motion, Malliavin calculus.

1. Introduction

The classical Mackey-Glass equation

$$\frac{dX_t}{dt} = \frac{\beta X_{t-r}}{1 + X_{t-r}^n} - \gamma X_t, \tag{1.1}$$

where β, n, r and γ are positive constants, was proposed by Mackey-Glass [1] as a model of hematopoiesis, X_t denotes the density of mature cells in blood circulation, and r is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream.

It is well known that the deterministic model (1.1) is popular in biology, but it is clear that this model will be more realistic if noise is added. Much of the literature has pointed out that fractional Brownian motion (or fractional noise) provide a natural theoretical framework to model many phenomena arising in biology (for example, see [2, 3]). If we choose to make the mortality rate, γ, as a stochastic parameter then we could model $\gamma(\omega)$ by

$$\gamma(\omega) = \gamma + \sigma \text{ ”fractional noise”}, \tag{1.2}$$

where $\gamma = E[\gamma(\omega)], \sigma$ are deterministic. Inserting (1.2) into Eq. (1.1), we get a stochastic version of the Mackey-Glass equation which has the form

$$dX_t = \left(\frac{\beta X_{t-r}}{1 + X_{t-r}^n} - \gamma X_t \right) dt + \sigma X_t dW_t^H \ , \ t \in [0, T], \tag{1.3}$$

$X_t = \phi(t)\ , \ t \in [-r, 0]$, where $\phi \in C[-r, 0], \phi(0) > 0$; where W_t^H is a fractional Brownian motion (fBm) of the Liouville form with Hurst index $H \in (\frac{1}{2}, 1)$.

Kyrtsou and Terraza's works recently (for instance, see [4, 5] and the references therein) have pointed out that the Mackey-Glass equation is able to be applied not only to biology, but also to finance when when some noise is added. More concretely, they considered the following model

$$X_{t+1} = X_t + \frac{\beta X_{t-r}}{1 + X_{t-r}^n} - \gamma X_t + \varepsilon_t, \tag{1.4}$$

where $n = 2$ and $\{\varepsilon_t\}$ is a heteroskedastic noise that is a sequence of the random variables with different variances. As the time interval decreases and becomes infinitesimal, Eq. (1.4) can be changed to one of its corresponding possible continuous-time forms that are described by

$$dX_t = \left(\frac{\beta X_{t-r}}{1 + X_{t-r}^n} - \gamma X_t \right) dt + \sigma X_t dW_t \ , \ t \in [0, T], \tag{1.5}$$

where W_t is a standard Brownian motion.

In the last few decades, the financial markets have been regarded as systems with complex behavior such as nonlinearity, fractal long-memory, and non-stationarity. A series of studies have been made to detect fractal long-memory in finance such as

- In the 1960s, Mandelbrot [6] found that the Hurst exponent H is 0.59 for Cotton prices.
- Peter [7] (1994) studied the distributional self-similarity of the Yen/Dollar exchange rates, from January 1, 1979 to September 30, 1992 and found that the Hurst exponent H is 0.64.

- Tabak and Cajueiro [8] (2005) discovered that interest rates with maturities between 6 and 24 months present increasing Hurst exponents from 0.425 to 0.575 and that interest rates with maturities between 7 and 20 years present monotonically increasing Hurst exponents, from 0.525 to 0.60, for daily observations on Japanese interest rates over the period from July 10, 1992, to July 7, 2004. Also, in [9] (2005) they studied the empirical evidence of long range dependence in returns and volatility for banking indices for 41 different countries and found that the Hurst exponents for developed markets and emerging markets, which are 0.526 and 0.578 using unshuffled returns and 0.525 and 0.581 using shuffled returns, respectively.
- Gao et al. [10] (2011) used detrended fluctuation analysis (DFA) to analyze the foreign exchange rate data between the US and Korea and found that the Hurst parameter is 0.57 ± 0.04

Therefore, it has been proposed from the empirical detections that one should replace the Brownian motion in the classical models by a process that has long-memory. In fact, in this field, many models have been made. For instance, in the fractional Black-Scholes model, the option pricing problems have been considered by Cutland et al. [11] and the optimal consumption and portfolio problems have been solved by Hu et al. [12].

In our context, it is interesting to replace Brownian motion in Eq. (1.5) by a fractional Brownian motion, then we one again obtain Eq. (1.3). Thus, the fractional stochastic Mackey-Glass processes seem to be a potential candidate to model objects in biology and finance. There will two fundamental questions that need answering:

(i) Do the fractional Mackey-Glass processes exist?

(ii) Which real-world's objects can be modeled as fractional stochastic Mackey-Glass processes?

In order to answer the second question we need to know about the numerical approximations for the solution of Eq. (1.3). In our opinion, this is not easy to do since the theory of stochastic delay differential equations is so new that only few works have appeared up to date. In fact, the approximations for (1.5) is now well established (see [13]), but for (1.3) has not yet been developed in even the simplest case. The aim of this paper is to find solution for the first question, i.e. to prove the existence and uniqueness of the solution of Eq. (1.3). Recently, a few authors (for example, see [14, 15]) have attempted to study the stochastic delay differential equations driven by fBm but unfortunately all of them do not cover (1.3). Moreover, their method cannot be used in this paper because our equation (1.3) contains X_t in fractional stochastic integral.

In the current work we use a definition of the stochastic integration with respect to fBm given by Carmona et al. [16] to study the equation (1.3) and introduce a new method that the existence and uniqueness of the solution are proved via an approximation equation driven by semimartingales.

This paper is organized as follows: In Section 2, we recall a semimartingale approximation of fBm and a definition of fractional stochastic integral given in [16]. Section 3 contains a main result of this paper that the semi-analytical solution of the equation (1.3) is found. Section 4 contains some comments.

2. Preliminaries

Let us recall some elements of the stochastic calculus of variations.

For $h \in L^2([0,T],\mathbb{R})$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)dW_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega,\mathcal{F},P)$ consisting of those classes of random variables of the form

$$F = f(W(h_1),...,W(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, L^2([0,T],\mathbb{R}))$, $h_1,...,h_n \in L^2([0,T],\mathbb{R})$. If F has the form (2.1), we define its derivative as the process $D^W F := \{D_t^W F, t \in [0,T]\}$ given by

$$D_t^W F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(W(h_1),...,W(h_n))h_k(t).$$

For any $p \geq 1$ we shall denote by $\mathbb{D}_W^{1,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \big[E|F|^p\big]^{\frac{1}{p}} + E\Big[\int_0^T |D_u^W F|^p du\Big]^{\frac{1}{p}}.$$

The fractional Brownian motion of the Liouville form with Hurst index $H \in (0,1)$ is a centered Gaussian process defined by

$$W_t^H = \int_0^t K(t,s)dW_s \tag{2.2}$$

where W is a standard Brownian motion and the kernel $K(t,s) = (t-s)^\alpha$, $\alpha = H - \frac{1}{2}$.

It is well known that we cannot use Itô calculus to analyze models with fractal noise because fBm is neither semimartingale nor Markov process, except for the case where Hurst index $H = \frac{1}{2}$. For $H > \frac{1}{2}$, the increments are positively correlated with the covariance at distance u decreases as u^{2H-2}, according to Beran's definition [17], fBm is a long memory process and presents an aggregation behavior. In case $H < \frac{1}{2}$, the increments are negatively correlated and fBm is intermittent.

For every $\varepsilon > 0$ we define

$$W_t^{H,\varepsilon} = \int_0^t K(t+\varepsilon, s)dW_s = \int_0^t (t-s+\varepsilon)^\alpha dW_s. \tag{2.3}$$

It is well known from [18, 19] that $W_t^{H,\varepsilon}$ is a semimartingale with the following decomposition

$$W_t^{H,\varepsilon} = \varepsilon^\alpha W_t + \int_0^t \varphi_s^\varepsilon ds, \tag{2.4}$$

where $\varphi_s^\varepsilon = \int\limits_0^s \alpha(s+\varepsilon-u)^{\alpha-1}dW_u$. Moreover, $W_t^{H,\varepsilon}$ converges to W_t^H in $L^p(\Omega)$, $p > 1$ uniformly in $t \in [0,T]$ as $\varepsilon \to 0$:

$$E|W_t^{H,\varepsilon} - W_t^H|^p \le c_{p,T}\varepsilon^{pH}.$$

It is well known from [16, 20] that for an adapted process f belonging to the space $\mathbb{D}_W^{1,2}$ we have

$$\int_0^t f_s\, dW_s^{H,\varepsilon} = \int_0^t f_s K(s+\varepsilon, s)\, dW_s + \int_0^t f_s \varphi_s^\varepsilon ds$$
$$= \int_0^t f_s K(t+\varepsilon, s)\, dW_s + \int_0^t \int_s^t (f_u - f_s)\, \partial_1 K(u+\varepsilon, s) du \delta W_s$$
$$+ \int_0^t \int_0^u D_s^W f_u\, \partial_1 K(u+\varepsilon, s) ds du\,, \quad (2.5)$$

where the second integral in the right-hand side is a Skorohod integral (we refer to [21] for more details about the Skorohod integral). In [20], Coutin has introduced the following hypothesis

Hypothesis (H): Assume that f is an adapted process belonging to the space $\mathbb{D}_W^{1,2}$ and that there exists β fulfilling $\beta + H > 1/2$ and $p > 1/H$ such that

(i) $\|f\|^2_{L^{1,2}_\beta} := \sup_{0<s<u<T} \frac{E\left[(f_u - f_s)^2 + \int_0^T (D_r^W f_u - D_r^W f_s)^2 dr\right]}{|u-s|^{2\beta}}$ is finite,

(ii) $\sup_{0<s<T} |f_s|$ belongs to $L^p(\Omega)$.

and proved that for $f \in$ **(H)**, $\int_0^t f_s\, dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$. Each term in the right-hand side of (2.5) converges to the same term where $\varepsilon = 0$. Then, it is "natural" to define

Definition 2.1. Let $f \in$ **(H)**. The fractional stochastic integral of f with respect to W^H is defined by

$$\int_0^t f_s\, dW_s^H = \int_0^t f_s K(t, s)\, dW_s + \int_0^t \int_s^t (f_u - f_s)\, \partial_1 K(u, s) du \delta W_s$$
$$+ \int_0^t du \int_0^u D_s^W f_u\, \partial_1 K(u, s) ds\,, \quad (2.6)$$

where $K(t, s) = (t-s)^\alpha, \partial_1 K(t, s) = \alpha(t-s)^{\alpha-1}$.

3. Main results

In order to prove the existence and uniqueness of the solution of (1.3) we use the method of steps as in classical delayed differential equation theory. We shall first prove the result for the interval $[0, r]$, then we use this solution process as the initial condition to solve the equation within the interval $[r, 2r]$, and so on.

Lemma 3.1. *Let $\{Z_t, t \in [0,T]\}$ be a centered Gaussian process with finite variance. Then*

$$\sup_{0\le t\le T} e^{\sigma Z_t} \in L^p(\Omega) \ \forall \ p > 1.$$

Proof. We refer to [22]. □

Theorem 3.1. *Suppose that Hurst index $H \in (\frac{1}{2}, 1)$. Let $\beta_t, t \in [0,T]$ be an adapted stochastic process such that*

$$\int_0^T \left(E|\beta_t|^{2+\delta} + \int_0^T E|D_u^W \beta_t|^{2+\delta} du \right) dt < \infty \text{ for some } \ \delta > 0. \tag{3.1}$$

Then the equation

$$dX_t = \big(\beta_t - \gamma X_t\big)dt + \sigma X_t dW_t^H \ , \ X_0 = x > 0 \tag{3.2}$$

has a unique solution which is given by

$$X_t = \Phi_t \left(X_0 + \int_0^t \beta_s (\Phi_s)^{-1} ds \right), \tag{3.3}$$

where $\Phi_t = \exp\big(-\lambda t + \sigma W_t^H\big)$. Moreover, this solution satisfies

$$\int_0^T \left(E|X_t|^{2+\frac{\delta}{2}} + \int_0^T E|D_u^W X_t|^{2+\frac{\delta}{2}} du \right) dt < \infty. \tag{3.4}$$

Proof. Before giving a proof, we want to emphasize that the study of the stochastic differential equations (SDEs) depends on the definitions of the stochastic integrals involved and in current work we use Definition 2.1. Up to now the existence and uniqueness of the solution are still an open problem in general. That is why we need to introduce a new method which is explained in detail as below.

From Definition 2.1, the solution of (3.2) is a stochastic process X_t belonging to **(H)** and satisfying

$$X_t = X_0 + \int_0^t \big(\beta_s - \gamma X_s\big) ds + \int_0^t \sigma X_s K(t,s)\, dW_s + \int_0^t \int_s^t \sigma (X_u - X_s)\, \partial_1 K(u,s) du \delta W_s + \int_0^t \int_0^u \sigma D_s^W X_u\, \partial_1 K(u,s) ds du. \quad (3.5)$$

Since (3.5) contains Skorohod differential and Malliavin derivative we can not directly apply the traditional methods to prove the existence and uniqueness of its solution. In order to avoid this difficult we consider the stochastic differential equation driven by semimartingales

$$dX_t^\varepsilon = \big(\beta_t - \gamma X_t^\varepsilon\big) dt + \sigma X_t^\varepsilon dW_t^{H,\varepsilon}\ ,\ X_0 = x > 0, \quad (3.6)$$

where $W_t^{H,\varepsilon}$ is defined by (2.3).

From the decomposition (2.4) we can rewrite (3.6) as follows

$$dX_t^\varepsilon = \big(\beta_t - \gamma X_t^\varepsilon + \sigma \varphi_t^\varepsilon X_t^\varepsilon\big) dt + \sigma \varepsilon^\alpha X_t^\varepsilon dW_t\ ,\ X_0 = x > 0.$$

Then it is easy to find a unique solution of above equation

$$X_t^\varepsilon = \Phi_t^\varepsilon \bigg(X_0 + \int_0^t \beta_s (\Phi_s^\varepsilon)^{-1} ds \bigg),$$

where

$$\Phi_t^\varepsilon = \exp\big(-(\lambda + \frac{1}{2}\varepsilon^{2\alpha}\sigma^2) t + \sigma W_t^{H,\varepsilon}\big).$$

Put $Y_t^\varepsilon = X_0 + \int_0^t \beta_s (\Phi_s^\varepsilon)^{-1} ds$, then by the chain rule of Malliavin derivative we have for any $u \leq t$

$$D_u^W X_t^\varepsilon = Y_t^\varepsilon D_u^W \Phi_t^\varepsilon + \Phi_t^\varepsilon D_u^W Y_t^\varepsilon,$$

where $D^W_u\Phi^\varepsilon_t, D^W_u Y^\varepsilon_t$ are well defined since

$$D^W_u\Phi^\varepsilon_t = \sigma(t-u+\varepsilon)^\alpha\Phi^\varepsilon_t, \tag{3.7}$$

$$\begin{aligned} D^W_u Y^\varepsilon_t &= \int\limits_u^t D^W_u[\beta_s(\Phi^\varepsilon_s)^{-1}]ds \\ &= \int\limits_u^t D^W_u[\beta_s](\Phi^\varepsilon_s)^{-1}ds - \int\limits_u^t \sigma\beta_s(s-u+\varepsilon)^\alpha(\Phi^\varepsilon_s)^{-1}ds. \end{aligned} \tag{3.8}$$

Thus $X^\varepsilon_t \in \mathbb{D}^{1,2}_W$. Consequently, by (2.5) Eq. (3.6) is equivalent to

$$\begin{aligned} X^\varepsilon_t = X_0 &+ \int\limits_0^t \big(\beta_s - \gamma X^\varepsilon_s\big)ds + \int\limits_0^t \sigma X^\varepsilon_s K(t+\varepsilon, s)\, dW_s \\ &+ \int\limits_0^t\int\limits_s^t \sigma(X^\varepsilon_u - X^\varepsilon_s)\,\partial_1 K(u+\varepsilon, s)du\delta W_s \\ &+ \int\limits_0^t\int\limits_0^u \sigma D^W_s X^\varepsilon_u\,\partial_1 K(u+\varepsilon, s)dsdu. \end{aligned} \tag{3.9}$$

In comparison (3.9) with (3.5) and taking account of the condition **(H)** we see that to prove X_t defined by (3.3) be the solution of (3.2) we need to show that $X^\varepsilon_t, X_t \in$ **(H)** and X^ε_t converges to X^ε_t in $L^2(\Omega)$ as $\varepsilon \to 0$.

We give some useful observations: the stochastic processes $\Phi^\varepsilon_t, (\Phi^\varepsilon_t)^{-1}$ and its Malliavin derivatives have any order finite moments. Indeed, we have

$$E|\Phi^\varepsilon_t|^p = \exp\big(-p(\lambda + \frac{1}{2}\varepsilon^{2\alpha}\sigma^2)t + \frac{1}{2}\sigma^2p^2\frac{(t+\varepsilon)^{2H}-\varepsilon^{2H}}{2H}\big),$$

$$E|(\Phi^\varepsilon_t)^{-1}|^p = \exp\big(p(\lambda + \frac{1}{2}\varepsilon^{2\alpha}\sigma^2)t + \frac{1}{2}\sigma^2p^2\frac{(t+\varepsilon)^{2H}-\varepsilon^{2H}}{2H}\big),$$

$$E|D^W_u\Phi^\varepsilon_t|^p \le \sigma^pT^{p\alpha}E|\Phi^\varepsilon_t|^p\ ,\ E|D^W_u\Phi^\varepsilon_t|^p \le \sigma^pT^{p\alpha}E|(\Phi^\varepsilon_t)^{-1}|^p.$$

Let X_1, X_2 be two Gaussian random variables with finite variances. Then for every $p > 1$ there exists a finite constant C such that

$$E|e^{X_1} - e^{X_2}|^p \leq C\bigg(E|X_1 - X_2|^{2p}\bigg)^{\frac{1}{2}}. \tag{3.10}$$

In the rest of this paper we use the following notation: C stands for a finite constant not depending on ε and whose value may vary from one occurrence to another.

Convergence: It is known from [19] that for $0 < \varepsilon < 1$, there exists a finite constant C depending only on p and T such that

$$E|\Phi_t^\varepsilon - \Phi_t|^p \leq C\varepsilon^{2p(H-\frac{1}{2})} \ \forall\ t \in [0, T].$$

Similarly, we have also

$$E|(\Phi_t^\varepsilon)^{-1} - (\Phi_t)^{-1}|^p \leq C\varepsilon^{2p(H-\frac{1}{2})} \ \forall\ t \in [0, T].$$

And then by applying the Hölder inequality we get

$$E|\Phi_t^\varepsilon(\Phi_s^\varepsilon)^{-1} - \Phi_t(\Phi_s)^{-1}|^p \leq C\varepsilon^{2p(H-\frac{1}{2})} \ \forall\ s, t \in [0, T].$$

Now

$$\begin{aligned} E|X_t^\varepsilon - X_t|^2 & \\ &\leq 2X_0 E|\Phi_t^\varepsilon - \Phi_t|^2 + 2\int_0^t E|\beta_s(\Phi_t^\varepsilon(\Phi_s^\varepsilon)^{-1} - \Phi_t(\Phi_s)^{-1})|^2 ds \\ &\leq 2X_0 E|\Phi_t^\varepsilon - \Phi_t|^2 \\ &+ 2\int_0^t \big(E|\beta_s|^{2+\frac{\delta}{2}}\big)^{\frac{\delta}{4+\delta}} \big(E|\Phi_t^\varepsilon(\Phi_s^\varepsilon)^{-1} - \Phi_t(\Phi_s)^{-1}|^{\frac{8+2\delta}{\delta}}\big)^{\frac{\delta}{4+\delta}} ds \\ &\leq C\varepsilon^{4(H-\frac{1}{2})}. \end{aligned} \tag{3.11}$$

The inequality (3.11) ensures that X_t^ε converges to X_t in $L^2(\Omega)$ as $\varepsilon \to 0$.

Check (H): For $0 \leq t_1 \leq t_2 \leq T$, using the inequality (3.10) we have

$$\begin{aligned} E|\Phi_{t_2}^\varepsilon - \Phi_{t_1}^\varepsilon|^p & \\ &\leq C\bigg(E|(\lambda + \frac{1}{2}\varepsilon^{2\alpha}\sigma^2)(t_2 - t_1) + \sigma(W_{t_2}^{H,\varepsilon} - W_{t_1}^{H,\varepsilon})|^{2p}\bigg)^{\frac{1}{2}} \\ &\leq C|t_2 - t_1|^{\frac{p}{2}}\ , \forall\ p > 1. \end{aligned} \tag{3.12}$$

(noting that $E|W^{H,\varepsilon}_{t_2} - W^{H,\varepsilon}_{t_1}|^2 \le |t_2 - t_1|$).

Applying Hölder we have

$$\begin{aligned} E|Y^\varepsilon_{t_2} - Y^\varepsilon_{t_1}|^{2+\frac{\delta}{2}} &= E\Big|\int_{t_1}^{t_2} \beta_s(\Phi^\varepsilon_s)^{-1} ds\Big|^{2+\frac{\delta}{2}} \\ &\le |t_2 - t_1|^{1+\frac{\delta}{2}} \int_{t_1}^{t_2} E|\beta_s(\Phi^\varepsilon_s)^{-1}|^{2+\frac{\delta}{2}} ds \\ &\le |t_2 - t_1|^{1+\frac{\delta}{2}} \int_0^T E|\beta_s(\Phi^\varepsilon_s)^{-1}|^{2+\frac{\delta}{2}} ds \le C|t_2 - t_1|^{1+\frac{\delta}{2}}. \quad (3.13) \end{aligned}$$

In particular, we have $E|Y^\varepsilon_t|^{2+\frac{\delta}{2}} \le X_0^{2+\frac{\delta}{2}} + CT^{1+\frac{\delta}{2}}\,\forall\, t \in [0,T]$. And then

$$\begin{aligned} E|X^\varepsilon_{t_2} - X^\varepsilon_{t_1}|^2 &\le 2E|\Phi^\varepsilon_{t_2}(Y^\varepsilon_{t_2} - Y^\varepsilon_{t_1})|^2 + 2E|Y^\varepsilon_{t_1}(\Phi^\varepsilon_{t_2} - \Phi^\varepsilon_{t_1})|^2 \\ &\le 2\big(E|\Phi^\varepsilon_{t_2}|^{\frac{8+2\delta}{\delta}}\big)^{\frac{\delta}{4+\delta}}\big(E|Y^\varepsilon_{t_2} - Y^\varepsilon_{t_1}|^{2+\frac{\delta}{2}}\big)^{\frac{1}{1+\frac{\delta}{4}}} \\ &\quad + 2\big(E|\Phi^\varepsilon_{t_2} - \Phi^\varepsilon_{t_1}|^{\frac{8+2\delta}{\delta}}\big)^{\frac{\delta}{4+\delta}}\big(E|Y^\varepsilon_{t_1}|^{2+\frac{\delta}{2}}\big)^{\frac{1}{1+\frac{\delta}{4}}} \\ &\le C\big(|t_2 - t_1|^{\frac{4+2\delta}{4+\delta}} + |t_2 - t_1|\big) \le C|t_2 - t_1|. \quad (3.14) \end{aligned}$$

From (3.7) and using a fundamental inequality $(a+b)^\gamma \le a^\gamma + b^\gamma; a, b > 0, \gamma \in (0,1)$ we get

$$E|D^W_u \Phi^\varepsilon_{t_2} - D^W_u \Phi^\varepsilon_{t_1}|^p \le C|t_2 - t_1|^{p\alpha}\ \forall\ p > 1.$$

Similarly proving (3.13) and (3.14) we have the following, respectively

$$\begin{aligned} E|D^W_u Y^\varepsilon_{t_2} - D^W_u Y^\varepsilon_{t_1}|^{2+\frac{\delta}{2}} &\le C|t_2 - t_1|^{1+\frac{\delta}{2}}, \\ E|D^W_u X^\varepsilon_{t_2} - D^W_u X^\varepsilon_{t_1}|^2 &\le C|t_2 - t_1|^{2\alpha}. \end{aligned} \quad (3.15)$$

Combining (3.14) and (3.15) we get

$$\|X^\varepsilon\|^2_{L^{1,2}_\beta} \le C \sup_{0<t_1<t_2<T} |t_2 - t_1|^{2\alpha - 2\beta},$$

which means that X^ε satisfies the condition (i) in **(H)** for any β such that $\frac{1}{2} - H < \beta \le \alpha = H - \frac{1}{2}$.

The condition (ii) in **(H)** directly follows from Corollary 3.1 and the following relation

$$\sup_{0\le t\le T}|X_t^\varepsilon| \le \Big(\sup_{0\le t\le T}\Phi_t^\varepsilon\Big)\Big(X_0+\int_0^T|\beta_s|(\Phi_s^\varepsilon)^{-1}ds\Big).$$

Thus, $X_t^\varepsilon \in$ (H).

Finally, the fact $X_t \in$ **(H)** is similarly proved and (3.4) is easy to check.

The proof of the theorem is complete. □

We now are position to state the main result of this paper.

Theorem 3.2. *The Mackey-Glass equation driven by fractional Brownian motion with $H > \frac{1}{2}$ admits a unique solution in* **(H)** *which can be explicitly found by the method of steps*

$$X_t = \phi(t), t \in [-r,0],$$

$$X_t = \Phi_t\Big(X_0+\int_0^t \tfrac{\beta X_{s-r}}{1+X_{s-r}^n}(\Phi_s)^{-1}ds\Big), t\in[0,T],$$

where

$$\Phi_t = \exp\big(-\lambda t+\sigma W_t^H\big).$$

Proof. Because the delay time, r, is discrete we can prove our theorem by the method of induction. For simplicity let us assume $T = Nr$, where N is a positive integer number. Our induction hypothesis, for $m < N$, is the following:

(H_m) The equation

$$X_t = \phi(0)+\int_0^t\Big(\frac{\beta X_{s-r}}{1+X_{s-r}^n}-\gamma X_s\Big)ds+\int_0^t\sigma X_s dW_s^H\ ,\ t\in[0,mr],$$

with $X_t = 0$, $t > mr$, has a unique solution in **(H)** *and this solution satisfies the condition (3.1) in Theorem 3.1 with some $\delta_m > 0$.*

Check (H_1). Let $t \in [0,r]$. Then $X_{t-r} = \phi(t-r)$ and the induction equation becomes

$$X_t = \phi(0) + \int_0^t \big(\beta_s - \gamma X_s\big)ds + \int_0^t \sigma X_s dW_s^H \ , \ t \in [0,r], \tag{3.16}$$

where $\beta_s = \frac{\beta\phi(s-r)}{1+\phi^n(s-r)}$.

It is obvious that β_s satisfies the condition (3.1) in Theorem 3.1 since ϕ is a continuous deterministic function. So (H_1) is true, the solution X_t satisfies (3.1) for any $\delta_1 > 0$.

Induction. Assume that (H_i) is true for $i \leq m$, with $m < N$. We wish to prove that (H_{m+1}) is also true. Consider the stochastic process defined as

$$V_t = \begin{cases} \phi(t-r) & \text{if } t \leq r, \\ X_{t-r} & \text{if } r < t \leq (m+1)r, \\ 0 & \text{if } t > (m+1)r \end{cases}$$

where X is the solution obtained in (H_m).

Put $\beta_s = \frac{\beta V_s}{1+V_s^n}$. Thus, for $t \in [0,(m+1)r]$, our problem has become the equation

$$X_t = \phi(0) + \int_0^t \big(\beta_s - \gamma X_s\big)ds + \int_0^t \sigma X_s dW_s^H \ , \ t \in [0,(m+1)r], \tag{3.17}$$

We need to check that β_s satisfies the condition (3.1) in Theorem 3.1. Consider the real function

$$f(x) = \frac{\beta x}{1+x^n}, x \geq 0.$$

We have

$$f'(x) = \frac{1+(1-n)x^n}{(1+x^n)^2}.$$

Hence, there exists a positive constant M such that

$$|f'(x)| \leq M \ \forall \ n > 0,$$

$$|f(x)| \leq \begin{cases} M & \text{if } n \geq 1 \\ \beta x^{1-n} & \text{if } 0 < n < 1. \end{cases}$$

As a consequence, β_s satisfies the condition (3.1) with $\delta = \delta_m$ since

$$\beta_s = f(V_s)\ ,\ D_u^W \beta_s = f'(V_s) D_u^W V_s.$$

Thus, (H_{m+1}) is true and the solution X_t satisfies (3.1) with $\delta_{m+1} = \frac{\delta_m}{2}$.

The proof of the theorem is complete. □

4. Conclusion and Possible extension

In this paper we proposed an approximation equation driven by semimartingales to prove the existence and uniqueness of the solution of a fractional stochastic Mackey-Glass equation. Our obtained result can be considered the first attempt to study fractional stochastic Mackey-Glass processes. There will still many open problems that need solving. Let us formulate some of them:

- As mentioned in Introduction, we need to construct the discrete-time approximation for the solution of Mackey-Glass equation (1.3) which plays important role in simulation.
- Determine the delay r and constants β and γ which are extremely important in the determination of dimensionality of the system.
- Find global stability conditions for the solution of Eq. (1.3) in terms of its coefficients.

Our method of approximation can be used to study a wider class of fractional stochastic differential equations with time delay. Indeed, for example, we consider the following equation as a generalization of the Mackey-Glass equation

$$\begin{aligned} dX_t = (b_1(t)X_t + b_2(t, X_{t-r}))dt \\ + (\sigma_1(t)X_t + \sigma_2(t, X_{t-r}))dW_t^H\ ,\ t \in [0,T], \end{aligned} \quad (4.1)$$

$X_t = \phi(t), t \in [-r, 0]; \phi \in C[-r, 0]$.

Similar to the previous section, we need to study the existence and uniqueness of the solution of the key equation

$$dX_t = (b_1(t)X_t + f(t))dt + (\sigma_1(t)X_t + g(t))dW_t^H\ ,\ t \in [0,T], \quad (4.2)$$

the initial condition X_0 is a real constant, where $b_1(t), \sigma_1(t)$ are some deterministic functions on $[0, T]$, and $f(t), g(t)$ are some adapted stochastic processes.

The semimartingale approximation equation corresponding to (4.2) is

$$dX_t^\varepsilon = (b_1(t)X_t^\varepsilon + f(t))dt + (\sigma_1(t)X_t^\varepsilon + g(t))dW_t^{H,\varepsilon}\ ,\ t \in [0, T] \tag{4.3}$$

and the approximation solution is given by

$$X_t^\varepsilon = \Phi_t^\varepsilon\bigg(X_0 + \int_0^t (f(s) - \varepsilon^{2\alpha}\sigma_1(s)g(s))(\Phi_s^\varepsilon)^{-1}ds + \int_0^t g(s)(\Phi_s^\varepsilon)^{-1}dW_s^{H,\varepsilon}\bigg),$$

where

$$\Phi_t^\varepsilon = \exp\bigg(\int_0^t (b_1(s) - \frac{1}{2}\varepsilon^{2\alpha}\sigma_1^2(s))ds + \int_0^t \sigma_1(s)dW_s^{H,\varepsilon}\bigg).$$

Under suitable conditions on the coefficients b_1, σ_1, f, g we can take the limit as $\varepsilon \to 0$ and then Eq. (4.2) has a unique solution which is given by

$$X_t = \Phi_t\bigg(X_0 + \int_0^t f(s)(\Phi_s)^{-1}ds + \int_0^t g(s)(\Phi_s)^{-1}dW_s^H\bigg), \tag{4.4}$$

where

$$\Phi_t = \exp\bigg(\int_0^t b_1(s)ds + \int_0^t \sigma_1(s)dW_s^H\bigg). \tag{4.5}$$

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